

VARIOUS NOTIONS OF POSITIVITY FOR BI-LINEAR MAPS AND APPLICATIONS TO TRI-PARTITE ENTANGLEMENT

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ABSTRACT. We consider bi-linear analogues of s -positivity for linear maps. The dual objects of these notions can be described in terms of Schmidt ranks for tri-tensor products and Schmidt numbers for tri-partite quantum states. These tri-partite versions of Schmidt numbers cover various kinds of bi-separability, and so we may interpret witnesses for those in terms of bi-linear maps. We give concrete examples of witnesses for various kinds of three qubit entanglement.

1. INTRODUCTION

Order structures are key ingredients in various subjects of mathematics as well as functional analysis, where linear maps preserving positivity play important roles. We call those positive linear maps. After representation theorem by Stinespring [24], complete positivity has been considered as a right morphism to study operator algebras, which are non-commutative in general. By definition of complete positivity, it is clear that there exist hierarchy structures between positivity and complete positivity, and this leads to define s -positivity of linear maps for natural numbers $s = 1, 2, \dots$. The notion of s -positivity turns out to be very useful in itself. For examples, various inequalities like Schwartz and Kadison inequalities in operator algebras already hold for 2-positive linear maps [4]. Importance of s -positivity is also recognized in current quantum information theory. Considering the dual objects of s -positive linear maps in matrix algebras, we get natural classification of bi-partite entanglement in terms of Schmidt ranks. See [10, 23, 26]. Furthermore, distillability problem which is one of the most important in quantum information theory can be formulated [8] in terms of Schmidt ranks.

The purpose of this note is to introduce the bi-linear analogues of s -positive linear maps, and classify tri-partite entanglement as dual objects. Our classification scheme include various kinds of bi-separability.

Positive linear maps and bi-partite entanglement are related through duality [10, 14], which goes back to the work by Woronowicz [28] in the seventies. Very recently, the second author [19] has shown that n -partite genuine separability is dual to positivity of $(n - 1)$ -linear maps whose linearization gives rise to positivity with respect to the

1991 *Mathematics Subject Classification.* 46L07, 81P15, 15A30, 46L05.

Key words and phrases. operator system, multi-linear maps, S -positive, Schmidt rank, entanglement, duality.

KHH and SHK were partially supported by NRF-2012R1A1A1012190 and NRFK grant 2013-004942, respectively.

function system maximal tensor product [9, 12] of matrix algebras. This means that it is necessary and sufficient to construct a positive multi-linear map, in order to detect entanglement which is not genuinely separable. A natural question arises: What kinds of positivity are suitable to detect another kinds of entanglement, for example, genuine entanglement which is not in the convex hull of bi-separable states with respect to all possible bi-partitions.

In the tri-partite cases, there are three kinds of bi-separability: $A-BC$, $B-CA$ and $C-AB$ separabilities according to bi-partitions. For a bi-linear map $\phi : M_A \times M_B \rightarrow M_C$ between matrix algebras to be dual objects of those notions, we need the following properties:

- (A) For each $x \in M_A^+$, the map $y \mapsto \phi(x, y)$ is completely positive.
- (B) For each $y \in M_B^+$, the map $x \mapsto \phi(x, y)$ is completely positive.
- (C) The linearization $M_A \otimes M_B \rightarrow M_C$ is positive with respect to the usual order.

In order to detect tri-partite genuine entanglement, we need a bi-linear map satisfying the above three conditions simultaneously. In this paper, we formulate the notions of positivity for bi-linear maps which explain the above three properties in a single framework.

It was shown in [17] that the linearization of a bi-linear map $\phi : \mathcal{S} \times \mathcal{T} \rightarrow \mathcal{R}$ between operator systems is completely positive with respect to the operator system maximal tensor products if and only if the following condition

$$(1) \quad [x_{i,j}] \in M_p(\mathcal{S})^+, [y_{k,\ell}] \in M_q(\mathcal{T})^+ \implies [\phi(x_{i,j}, y_{k,\ell})] \in M_{pq}(\mathcal{R})^+$$

holds for every $p, q = 1, 2, \dots$. It is tempting to call the property (1) as (p, q) -positivity. Then ϕ satisfies the condition (A) if and only if it is $(1, \infty)$ -positive, that is, $(1, q)$ -positive for every $q = 1, 2, \dots$. Condition (B) is, of course, nothing but $(\infty, 1)$ -positivity. But, it is not possible to formulate the condition (C) with (p, q) -positivity. In fact, there is no way to control the matrix size over the range spaces with the above condition (1). This motivates the following definition.

Definition 1.1. Let $S = (s_1, s_2, \dots, s_n)$ be an n -tuple of natural numbers. An $(n-1)$ -linear map $\phi : \mathcal{S}_1 \times \dots \times \mathcal{S}_{n-1} \rightarrow \mathcal{S}_n$ between operator systems is said to be S -positive if the following condition holds:

$$x_k = [x_{i_k, j_k}^k] \in M_{s_k}(\mathcal{S}_k)^+ \text{ for } k = 1, 2, \dots, n-1 \text{ and } \alpha \in M_{s_n, s_1 \dots s_{n-1}} \\ \implies \alpha[\phi(x_{i_1, j_1}^1, \dots, x_{i_{n-1}, j_{n-1}}^{n-1})]\alpha^* \in M_{s_n}(\mathcal{S}_n)^+$$

Then the above conditions (A), (B) and (C) become $(1, \infty, \infty)$, $(\infty, 1, \infty)$ and $(\infty, \infty, 1)$ -positivity, respectively. See Proposition 3.3. In the case of linear maps with $n = 2$, it is (p, q) -positive if and only if it is $(p \wedge q, p \wedge q)$ -positive if and only if it is $p \wedge q$ -positive in the usual sense, where $p \wedge q$ denotes the minimum of p and q . Therefore, the above definition reduces to the usual notion of s -positivity in the case

of linear maps. Furthermore, a bi-linear map satisfies the condition (1) if and only if it is (p, q, pq) -positive.

If we restrict ourselves in the cases when domains and ranges are matrix algebras, then we can consider the Choi matrices of bi-linear maps. We find conditions when they are positive, that is, positive semi-definite. In the course of discussion, we get bi-linear version of the isomorphism between completely positive linear maps and positive block matrices, as well as decomposition of completely positive maps into the sum of elementary operators [5, 18]. The linearization of (p, q, r) -positive bi-linear maps will be described in terms of suitable quantizations of domains and ranges.

We introduce the notion that Schmidt numbers for tri-partite states ϱ are less than or equal to triplets (p, q, r) of natural numbers. We write this property by $\text{SN}(\varrho) \leq (p, q, r)$. To do this, we first define the Schmidt ranks for vectors in the tensor products of three vector spaces. For this purpose, we use the natural isomorphisms between tensor products and linear mapping spaces, and consider the dimensions of supports and ranges of the corresponding maps. We establish the duality between (p, q, r) -positive bi-linear maps and states ϱ with the property $\text{SN}(\varrho) \leq (p, q, r)$.

After we summarize briefly in the next section several notions in operator systems we need, we present in Section 3 properties of (p, q, r) -positive bi-linear maps mentioned above. We give the definition of $\text{SN}(\varrho) \leq (p, q, r)$ in Section 4, and prove the duality in Section 5, where we also interpret the notion (p, q, r) -positivity in various ways. We exhibit in Section 6 concrete examples of bi-linear maps with various kinds of positivity in two dimensional matrix algebras. They will be witnesses for various kinds of three qubit entanglement, including witnesses for genuine entanglement. We close this paper to discuss several related problems in the last section.

Throughout this note, we will use notations $\mathcal{H}_A, \mathcal{H}_B$ and \mathcal{H}_C for complex Hilbert spaces $\mathbb{C}^a, \mathbb{C}^b$ and \mathbb{C}^c , respectively. Matrix algebras on them will be also denoted by M_A, M_B and M_C , and so they are $a \times a, b \times b$ and $c \times c$ matrix algebras, respectively. For a bi-linear map $\phi : \mathcal{S} \times \mathcal{T} \rightarrow \mathcal{R}$, we denote by $\tilde{\phi} : \mathcal{S} \otimes \mathcal{T} \rightarrow \mathcal{R}$ its linearization which sends $x \otimes y$ to $\phi(x, y)$. For $x = [x_{i,j}] \in M_p(\mathcal{S})$ and $y = [y_{k,\ell}] \in M_q(\mathcal{T})$, we write $\phi_{p,q}(x, y) = [\phi(x_{i,j}, y_{k,\ell})] \in M_{pq}(\mathcal{R})$ for notational convenience. If $\phi : \mathcal{S} \rightarrow \mathcal{R}$ is a linear map then we also write $\phi_p(x) = [\phi(x_{i,j})] \in M_p(\mathcal{R})$. Recall that ϕ is p -positive if $x \in M_p(\mathcal{S})^+$ implies $\phi_p(x) \in M_p(\mathcal{R})^+$.

The authors are grateful to Kil-Chan Ha and Jaeseong Heo for fruitful discussion on the topics. This work has been completed when the second author was visiting Singapore. He is also grateful to Denny Leung and Wai Shing Tang for their warm hospitality and stimulating discussion during his stay.

2. TENSOR PRODUCT AND QUANTIZATION OF OPERATOR SYSTEMS

A unital self-adjoint space of bounded operators on a Hilbert space is said to be an operator system. If \mathcal{S} is an operator system acting on a Hilbert space \mathcal{H} , then the

space $M_n(\mathcal{S})$ of all $n \times n$ matrices over \mathcal{S} acts on the Hilbert space \mathcal{H}^n . The order structures of $M_n(\mathcal{S})$ for each $n \in \mathbb{N}$ are related by the following compatibility relation:

$$(2) \quad x \in M_n(\mathcal{S})^+, \alpha \in M_{m,n} \implies \alpha x \alpha^* \in M_m(\mathcal{S})^+.$$

The identity operator $\text{id}_{\mathcal{H}^n}$ on the Hilbert space \mathcal{H}^n plays the role of order unit of $M_n(\mathcal{S})$, that is, for every self-adjoint $x \in M_n(\mathcal{S})$ there is $r > 0$ such that $x \leq r \cdot \text{id}_{\mathcal{H}^n}$. This order unit also has the Archimedean property: If $x \in M_n(\mathcal{S})$ and $\varepsilon \cdot \text{id}_{\mathcal{H}^n} + x \geq 0$ for each $\varepsilon > 0$ then $x \in M_n(\mathcal{S})^+$.

If V is a $*$ -vector space and C_n is a cone in $M_n(V)_h$ satisfying (2), then we call $\{C_n\}_{n=1}^\infty$ a matrix ordering and $(V, \{C_n\}_{n=1}^\infty)$ a matrix ordered $*$ -vector space. Choi and Effros [6] showed that matrix ordered $*$ -vector spaces equipped with Archimedean matrix order units can be realized as unital self-adjoint spaces of bounded operators on a Hilbert space. Thus, we also call them operator systems. An operator system structure of \mathcal{S} determines an operator space structure on \mathcal{S} . Especially, if x is a Hermitian element of $M_n(\mathcal{S})$ then we have

$$\|x\|_n = \inf\{r > 0 : -rI_n \otimes 1_{\mathcal{S}} \leq x \leq rI_n \otimes 1_{\mathcal{S}}\}.$$

We proceed to recall the definition [17] of the maximal tensor product of operator systems. For two operator systems \mathcal{S} and \mathcal{T} , the sets

$$D_n^{\max}(\mathcal{S}, \mathcal{T}) = \{\alpha(P \otimes Q)\alpha^* : P \in M_k(\mathcal{S})^+, Q \in M_\ell(\mathcal{T})^+, \alpha \in M_{n,k\ell}, k, \ell \in \mathbb{N}\}$$

for each $n = 1, 2, \dots$ give rise to a matrix ordering on $\mathcal{S} \otimes \mathcal{T}$ with a matrix order unit $1_{\mathcal{S}} \otimes 1_{\mathcal{T}}$. Let $\{M_n(\mathcal{S} \otimes_{\max} \mathcal{T})^+\}_{n=1}^\infty$ be the Archimedeanization of the matrix ordering $\{D_n^{\max}(\mathcal{S}, \mathcal{T})\}_{n=1}^\infty$. Then it can be written as

$$(3) \quad M_n(\mathcal{S} \otimes_{\max} \mathcal{T})^+ = \{z \in M_n(\mathcal{S} \otimes \mathcal{T}) : \forall \varepsilon > 0, z + \varepsilon I_n \otimes 1_{\mathcal{S}} \otimes 1_{\mathcal{T}} \in D_n^{\max}(\mathcal{S}, \mathcal{T})\}.$$

We call the operator system $(\mathcal{S} \otimes \mathcal{T}, \{M_n(\mathcal{S} \otimes_{\max} \mathcal{T})^+\}_{n=1}^\infty, 1_{\mathcal{S}} \otimes 1_{\mathcal{T}})$ the maximal tensor product of \mathcal{S} and \mathcal{T} , and denote by $\mathcal{S} \otimes_{\max} \mathcal{T}$. The family $\{M_n(\mathcal{S} \otimes_{\max} \mathcal{T})^+\}_{n=1}^\infty$ is the smallest among positive cones of operator system structures on $\mathcal{S} \otimes \mathcal{T}$. For unital C^* -algebras \mathcal{A} and \mathcal{B} , we have the completely order isomorphic inclusion $\mathcal{A} \otimes_{\max} \mathcal{B} \subset \mathcal{A} \otimes_{C^* \max} \mathcal{B}$. In particular, $M_m \otimes_{\max} M_n \simeq M_{mn}$ is a complete order isomorphism. On the other hand, one can also define the largest positive cones on $\mathcal{S} \otimes \mathcal{T}$ to get the minimal tensor product $\mathcal{S} \otimes_{\min} \mathcal{T}$.

For given Archimedean order unit spaces V , there are two canonical ways to endow matrix order structures with which they are operator systems [20]. These processes are usually called quantization. One way is to endow the largest positive cones of operator system structures on V whose first level positive cone coincides with V^+ , to get the minimal operator system $\text{OMIN}(V)$. The other is to endow the smallest positive cones, to get the the maximal operator system $\text{OMAX}(V)$.

For a given operator system \mathcal{S} and a natural number $k \in \mathbb{N}$, one can also define two operator systems, super k -maximal operator systems $\text{OMAX}^k(\mathcal{S})$ and super k -minimal

operator systems $\text{OMIN}^k(\mathcal{S})$, respectively [29]. For each $n, k \in \mathbb{N}$, we set

$$D_n^{\max, k} = \left\{ \alpha \begin{pmatrix} s_1 & & \\ & \ddots & \\ & & s_m \end{pmatrix} \alpha^* : \alpha \in M_{n, mk}, s_i \in M_k(\mathcal{S})^+, m \in \mathbb{N} \right\}.$$

Applying Archimedeanization process, we get

$$(4) \quad C_n^{\max, k} = \{s \in M_n(\mathcal{S}) : \forall \varepsilon > 0, x + \varepsilon I_n \otimes 1_{\mathcal{S}} \in D_n^{\max, k}(\mathcal{S})\}.$$

Then, $(\mathcal{S}, \{C_k^{\max}(\mathcal{S})\}_{k=1}^{\infty}, 1_{\mathcal{S}})$ is an operator system which is denoted by $\text{OMAX}^k(\mathcal{S})$. In particular, we have $\text{OMAX}^1(\mathcal{S}) = \text{OMAX}(\mathcal{S})$. The family $\{C_k^{\max}(\mathcal{S})\}_{k=1}^{\infty}$ is the smallest among positive cones of operator system structures on \mathcal{S} whose k -th level positive cones coincide with $M_k(\mathcal{S})^+$. A linear map $\phi : \text{OMAX}^k(\mathcal{S}) \rightarrow \mathcal{T}$ is k -positive if and only if it is completely positive. Moreover, this property characterizes $\text{OMAX}^k(\mathcal{S})$ [29, Theorem 4.6].

For each $n, k \in \mathbb{N}$, we set

$$C_n^{\min, k} = \{[s_{i,j}] \in M_n(\mathcal{S}) : [\varphi(s_{i,j})] \geq 0 \text{ for each } \varphi \in S_k(\mathcal{S}), \},$$

where $S_k(\mathcal{S})$ denotes the set of unital completely positive linear maps from \mathcal{S} into M_k . Then, $(\mathcal{S}, \{C_k^{\min}(\mathcal{S})\}_{k=1}^{\infty}, 1_{\mathcal{S}})$ is an operator system which is denoted by $\text{OMIN}^k(\mathcal{S})$. In particular, we have $\text{OMIN}^1(\mathcal{S}) = \text{OMIN}(\mathcal{S})$. The family $\{C_k^{\min}(\mathcal{S})\}_{k=1}^{\infty}$ is the largest among positive cones of operator system structures on \mathcal{S} whose k -th level positive cones coincide with $M_k(\mathcal{S})^+$. A linear map $\phi : \mathcal{S} \rightarrow \text{OMIN}^k(\mathcal{T})$ is k -positive if and only if it is completely positive. Moreover, this property characterizes $\text{OMIN}^k(\mathcal{T})$ [29, Theorem 3.7].

Duals of operator systems are matrix ordered by the cones

$$M_n(\mathcal{S}^*)^+ = \text{CP}(\mathcal{S}, M_n), \quad n = 1, 2, \dots,$$

where $\text{CP}(\mathcal{S}, M_n)$ denotes the set of all completely positive linear maps from \mathcal{S} into M_n . With this matrix ordering, we have the complete order isomorphism [17, Lemma 5.7, Theorem 5.8]

$$(5) \quad (\mathcal{S} \otimes_{\max} \mathcal{T})^* \simeq \mathcal{L}(\mathcal{S}, \mathcal{T}^*),$$

where $\mathcal{L}(\mathcal{S}, \mathcal{T}^*)$ is matrix ordered by

$$M_n(\mathcal{L}(\mathcal{S}, \mathcal{T}^*))^+ = \text{CP}(\mathcal{S}, M_n(\mathcal{T}^*)).$$

Unfortunately, duals of operator systems fail to be operator systems in general due to the lack of matrix order unit. However, duals of matrix algebras are again operator systems because the trace satisfies the condition of Archimedean matrix order unit. Moreover, matrix algebras are self-dual operator systems: Every $x \in M_n$ corresponds to $f_x \in M_n^*$ given by $f_x(y) = \text{Tr}(xy^t) = \sum_{i,j=1}^n x_{i,j}y_{i,j}$. The map

$$\gamma : x \in M_n \mapsto f_x \in M_n^*$$

is a unital complete order isomorphism [20, Theorem 6.2]. Related with quantization, γ gives rise to the duality [29, Proposition 6.5]:

$$(6) \quad \text{OMAX}^k(M_n) \simeq \text{OMIN}^k(M_n)^*, \quad \text{OMIN}^k(M_n) \simeq \text{OMAX}^k(M_n)^*.$$

We will use γ to define the dual map of a bi-linear map from $M_A \times M_B$ into M_C which is given by a permutation on $\{A, B, C\}$.

3. S -POSITIVE BI-LINEAR MAPS

Following proposition shows that some combinations of numbers in the definition of (p, q, r) -positivity are redundant.

Proposition 3.1. *Suppose that $\phi : \mathcal{S} \times \mathcal{T} \rightarrow \mathcal{R}$ is a bi-linear map in operator systems \mathcal{S}, \mathcal{T} and \mathcal{R} . For $p, q = 1, 2, \dots$, the following are equivalent:*

- (i) ϕ satisfies the condition (1);
- (ii) ϕ is (p, q, r) -positive for each $r = 1, 2, \dots$;
- (iii) ϕ is (p, q, r) -positive for some $r \geq pq$;
- (iv) ϕ is (p, q, pq) -positive.

Proof. The implication (i) \implies (ii) follows from the relation (2), and (ii) \implies (iii) is clear. For the direction (iii) \implies (iv), we note that

$$\begin{pmatrix} \alpha \phi_{p,q}(x, y) \alpha^* & 0 \\ 0 & 0_{r-pq} \end{pmatrix} = \begin{pmatrix} \alpha & \\ & 0_{r-pq, pq} \end{pmatrix} \phi_{p,q}(x, y) \begin{pmatrix} \alpha^* & 0_{pq, r-pq} \end{pmatrix} \in M_r(\mathcal{R})^+$$

for $x \in M_p(\mathcal{S})^+, y \in M_q(\mathcal{T})^+$ and $\alpha \in M_{pq}$. This implies that $\alpha \phi_{p,q}(x, y) \alpha^* \in M_{pq}(\mathcal{S})^+$, as it was required. Finally, we take $\alpha = I_{pq}$ for (iv) \implies (i). \square

Proposition 3.2. *Suppose that $\phi : \mathcal{S} \times \mathcal{T} \rightarrow \mathcal{R}$ is a bi-linear map in operator systems \mathcal{S}, \mathcal{T} and \mathcal{R} . We have the following:*

- (i) ϕ is $(1, q, r)$ -positive if and only if ϕ is $(1, q \wedge r, q \wedge r)$ -positive.
- (ii) ϕ is $(p, 1, r)$ -positive if and only if ϕ is $(p \wedge r, 1, p \wedge r)$ -positive.

Proof. Since we may exchange the role of \mathcal{S} and \mathcal{T} , it suffices to prove (i). This is immediate when $q \leq r$ by Proposition 3.1. Let $q \geq r$ and $x \in \mathcal{S}^+$. If ϕ is $(1, q, r)$ -positive then we have

$$\alpha \phi_{1,r}(x, y) \alpha^* = \begin{pmatrix} \alpha & 0_{r, q-r} \end{pmatrix} \phi_{1,q}(x, y \oplus 0_{q-r}) \begin{pmatrix} \alpha^* \\ 0_{q-r, r} \end{pmatrix} \in M_r(\mathcal{R})^+,$$

for $y \in M_r(\mathcal{T})^+$ and $\alpha \in M_r$, and so ϕ is $(1, r, r)$ -positive. For the converse, suppose that ϕ is $(1, r, r)$ -positive. Then we have

$$\alpha \phi_{1,q}(x, y) \alpha^* = \phi_{1,r}(x, \alpha y \alpha^*) \in M_r(\mathcal{R})^+$$

for $y \in M_q(\mathcal{T})^+$ and $\alpha \in M_{r,q}$. This shows that ϕ is $(1, q, r)$ -positive. \square

Taking $\mathcal{S} = \mathbb{C}$ in Proposition 3.2 (i), we see that a linear map is (q, r) -positive if and only if it is $(q \wedge r, q \wedge r)$ -positive if and only if it is $q \wedge r$ -positive in the usual

sense. When \mathcal{S}, \mathcal{T} and \mathcal{R} are matrix algebras, we will see later that the role of p, q and r in Propositions 3.1 and 3.2 may be permuted together with $\mathcal{S}, \mathcal{T}, \mathcal{R}$. See Corollary 5.4. If one of p, q is 1 then we may assume that the others coincide by Proposition 3.2. These are the most important cases with which conditions (A), (B) and (C) discussed in Introduction may be explained.

Proposition 3.3. *Suppose that $\phi : \mathcal{S} \times \mathcal{T} \rightarrow \mathcal{R}$ is a bi-linear map in operator systems \mathcal{S}, \mathcal{T} and \mathcal{R} . We have the following:*

- (i) ϕ is $(1, p, p)$ -positive if and only if the map $y \mapsto \phi(x, y)$ is p -positive for each $x \in \mathcal{S}^+$.
- (ii) ϕ is $(p, 1, p)$ -positive if and only if the map $x \mapsto \phi(x, y)$ is p -positive for each $y \in \mathcal{T}^+$.
- (iii) ϕ is $(p, p, 1)$ -positive if and only if $\sum_{i,j=1}^p \phi(x_{i,j}, y_{i,j}) \in \mathcal{R}^+$ for each $x \in M_p(\mathcal{S})^+$ and $y \in M_p(\mathcal{T})^+$.
- (iv) When $\mathcal{S} = M_p$, ϕ is $(p, p, 1)$ -positive if and only if $\tilde{\phi} : M_p(\mathcal{T}) \rightarrow \mathcal{R}$ is positive.

Proof. Statements (i) and (ii) follow immediately from Proposition 3.1.

(iii). We denote by $\{e_i\}_{i=1}^p$ the canonical basis of \mathbb{C}^p written as column vectors. Then the identity

$$\sum_{i,j=1}^p \phi(x_{i,j}, y_{i,j}) = \begin{pmatrix} e_1^t & \cdots & e_p^t \end{pmatrix} \phi_{p,p}(x, y) \begin{pmatrix} e_1 \\ \vdots \\ e_p \end{pmatrix}$$

shows that if ϕ is $(p, p, 1)$ -positive then $\sum_{i,j=1}^p \phi(x_{i,j}, y_{i,j}) \in \mathcal{R}^+$ whenever $x \in M_p(\mathcal{S})^+$ and $y \in M_p(\mathcal{T})^+$.

For the other direction, let $x \in M_p(\mathcal{S})^+, y \in M_p(\mathcal{T})^+$ and $\alpha \in M_{1,p^2}$. If we denote by $\tilde{\alpha}$ the $p \times p$ matrix whose entries are given by $\tilde{\alpha}_{ij} = \alpha_{1,(i-1)p+j}$, then we have

$$\alpha = \begin{pmatrix} e_1^t & \cdots & e_p^t \end{pmatrix} \begin{pmatrix} \tilde{\alpha} & & \\ & \ddots & \\ & & \tilde{\alpha} \end{pmatrix} = \begin{pmatrix} e_1^t & \cdots & e_p^t \end{pmatrix} (I_p \otimes \tilde{\alpha}).$$

Therefore, we have

$$\begin{aligned} \alpha \phi_{p,p}(x, y) \alpha^* &= \begin{pmatrix} e_1^t & \cdots & e_p^t \end{pmatrix} (I_p \otimes \tilde{\alpha}) \phi_{p,p}(x, y) (I_p \otimes \tilde{\alpha})^* \begin{pmatrix} e_1^t & \cdots & e_p^t \end{pmatrix}^* \\ &= \begin{pmatrix} e_1^t & \cdots & e_p^t \end{pmatrix} (I_p \otimes \tilde{\alpha}) \tilde{\phi}_{p^2}(x \otimes y) (I_p \otimes \tilde{\alpha})^* \begin{pmatrix} e_1^t & \cdots & e_p^t \end{pmatrix}^* \\ &= \begin{pmatrix} e_1^t & \cdots & e_p^t \end{pmatrix} \phi_{p,p}(x, \tilde{\alpha} y \tilde{\alpha}^*) \begin{pmatrix} e_1^t & \cdots & e_p^t \end{pmatrix}^*. \end{aligned}$$

If we write $z = \tilde{\alpha} y \tilde{\alpha}^* \in M_n(\mathcal{T})^+$ then this is nothing but $\sum_{i,j=1}^p \phi(x_{i,j}, z_{i,j}) \in \mathcal{R}^+$ by assumption, as it was required.

(iv). Suppose that $\phi : M_p \times \mathcal{T} \rightarrow \mathcal{R}$ is $(p, p, 1)$ -positive, and $y \in M_p(\mathcal{T})^+$. Since $[e_{i,j}]_{i,j} \in M_p(M_p)^+$, we have

$$\tilde{\phi}(y) = \tilde{\phi} \left(\sum_{i,j=1}^p e_{i,j} \otimes y_{i,j} \right) = \sum_{i,j=1}^p \tilde{\phi}(e_{i,j}, y_{i,j}) \in \mathcal{R}^+$$

by (iii). For the converse, suppose that $\tilde{\phi} : M_p(\mathcal{T}) \rightarrow \mathcal{R}$ is positive. We note that

$$\alpha(x \otimes y)\alpha^* \in (M_p \otimes_{\max} \mathcal{T})^+ = M_p(\mathcal{T})^+,$$

for $x \in M_p(M_p)^+$, $y \in M_p(\mathcal{T})^+$ and $\alpha \in M_{1,p^2}$. It follows that

$$\alpha\phi_{p,p}(x, y)\alpha^* = \alpha\tilde{\phi}_{p^2}(x \otimes y)\alpha^* = \tilde{\phi}(\alpha(x \otimes y)\alpha^*) \in \mathcal{R}^+,$$

which shows that ϕ is $(p, p, 1)$ -positive. \square

Now, we consider bi-linear maps $\phi : M_A \times M_B \rightarrow M_C$ between matrix algebras M_A , M_B and M_C . In this case, a bi-linear map may be described in terms of associated Choi matrix, as it was defined in [19] for multi-linear cases. For a given bi-linear map $\phi : M_A \times M_B \rightarrow M_C$, the Choi matrix C_ϕ is defined by

$$C_\phi = \sum_{i,j=1}^a \sum_{k,\ell=1}^b |i\rangle\langle j| \otimes |k\rangle\langle \ell| \otimes \phi(|i\rangle\langle j|, |k\rangle\langle \ell|) \in M_A \otimes M_B \otimes M_C.$$

For a given matrix $C \in M_A \otimes M_B \otimes M_C$, we may write

$$\begin{aligned} C &= \sum_{i,j=1}^a |i\rangle\langle j| \otimes C_{i,j} \in M_A \otimes (M_B \otimes M_C) \\ &= \sum_{i,j=1}^a \sum_{k,\ell=1}^b |i\rangle\langle j| \otimes |k\rangle\langle \ell| \otimes C_{(i,k),(j,\ell)} \in M_A \otimes M_B \otimes M_C. \end{aligned}$$

We associate the bi-linear map $\phi_C : M_A \otimes M_B \rightarrow M_C$ by

$$\phi_C(|i\rangle\langle j|, |k\rangle\langle \ell|) = C_{(i,k),(j,\ell)} \in M_C.$$

The correspondences $\phi \mapsto C_\phi$ and $C \mapsto \phi_C$ are just the Choi-Jamiołkowski isomorphisms [5, 15] when $M_B = \mathbb{C}$.

We consider the elementary bi-linear map $\phi_V : M_A \times M_B \rightarrow M_C$ with an $c \times ab$ matrix V , defined by

$$(7) \quad \phi_V(x, y) = V(x \otimes y)V^*, \quad x \in M_A, y \in M_B.$$

It is obvious that the map ϕ_V satisfies the condition (1) for every $p, q = 1, 2, \dots$, and so it is (p, q, r) -positive for every $p, q, r = 1, 2, \dots$ by Proposition 3.1. To calculate its Choi matrix, we write

$$|V_{(i,k)}\rangle = V|i\rangle|k\rangle \in \mathcal{H}_C,$$

which is the (i, k) -th column of the $c \times ab$ matrix V . Then we see that

$$\begin{aligned}
C_{\phi_V} &= \sum_{i,j=1}^a \sum_{k,\ell=1}^b |i\rangle\langle j| \otimes |k\rangle\langle\ell| \otimes V(|i\rangle\langle j| \otimes |k\rangle\langle\ell|) V^* \in M_A \otimes M_B \otimes M_C \\
&= \sum_{i,j=1}^a \sum_{k,\ell=1}^b |i\rangle\langle k| \langle j| \langle\ell| \otimes |V_{(i,k)}\rangle \langle V_{(j,\ell)}| \in (M_A \otimes M_B) \otimes M_C \\
&= \left(\sum_{(i,k)=(1,1)}^{(a,b)} |i\rangle\langle k| |V_{(i,k)}\rangle \right) \left(\sum_{(j,\ell)=(1,1)}^{(a,b)} \langle j| \langle\ell| \langle V_{(j,\ell)}| \right) \in M_A \otimes M_B \otimes M_C
\end{aligned}$$

is a positive matrix of rank one whose range vector is given by $\sum_{(i,k)=(1,1)}^{(a,b)} |i\rangle\langle k| |V_{(i,k)}\rangle$. Conversely, If $C_\phi \in M_A \otimes M_B \otimes M_C$ is positive with rank one then ϕ is of the form in (7), where V is given by the above relation in the obvious way. This actually proves the equivalence between statements (v) and (vi) in the following:

Theorem 3.4. *For a bi-linear map $\phi : M_A \times M_B \rightarrow M_C$, the following are equivalent:*

- (i) ϕ is (p, q, r) -positive for each $p, q, r = 1, 2, \dots$;
- (ii) ϕ is (a, b, ab) -positive;
- (iii) ϕ satisfies the condition (1) for each $p, q = 1, 2, \dots$;
- (iv) ϕ satisfies the condition (1) with $p = a$ and $q = b$;
- (v) the Choi matrix C_ϕ is positive;
- (vi) $\phi = \sum \phi_{V_i}$ with $c \times ab$ matrices V_i 's.

Proof. Equivalences (i) \iff (iii) and (ii) \iff (iv) come from Proposition 3.1. We proceed to prove the implications (iv) \implies (v) \implies (vi) \implies (iii). The condition (iv) tells us that if $\sum_{i,j=1}^a |i\rangle\langle j| \otimes x_{i,j} \in (M_a \otimes M_A)^+$ and $\sum_{k,\ell=1}^b |k\rangle\langle\ell| \otimes y_{k,\ell} \in (M_b \otimes M_B)^+$ then

$$\sum_{i,j=1}^a \sum_{k,\ell=1}^b |i\rangle\langle k| \langle j| \langle\ell| \otimes \phi(x_{i,j}, y_{k,\ell}) \in (M_{ab} \otimes M_C)^+.$$

This implies that the Choi matrix C_ϕ is positive because both $\sum_{i,j=1}^a |i\rangle\langle j| \otimes |i\rangle\langle j|$ and $\sum_{k,\ell=1}^b |k\rangle\langle\ell| \otimes |k\rangle\langle\ell|$ are positive. Therefore, we see that (iv) implies (v). If C_ϕ is positive then it is the sum of rank one positive matrices by the spectral decomposition, and so we see that ϕ is of the form in (vi) by the above discussion. It is easy to see that the bi-linear map ϕ_V satisfies the condition (iii). \square

The Hadamard product $[x_{i,j}] \circ [y_{i,j}] = [x_{i,j} y_{i,j}]$ between $n \times n$ matrices is a typical example of a bi-linear map satisfying the conditions in Theorem 3.4. Its Choi matrix is given by

$$\sum_{i,j=0}^{n-1} |i\rangle\langle j| \otimes |i\rangle\langle j| \otimes |i\rangle\langle j| \in M_n \otimes M_n \otimes M_n,$$

which is the rank one positive matrix onto the vector $\sum_{i=0}^{n-1} |i\rangle|i\rangle|i\rangle \in \mathbb{C}^{n^3}$. We close this section with the linearization of (p, q, r) -positive bi-linear maps.

Theorem 3.5. Suppose that $\phi : \mathcal{S} \times \mathcal{T} \rightarrow \mathcal{R}$ is a bi-linear map for operator systems \mathcal{S}, \mathcal{T} and \mathcal{R} . Then the following are equivalent:

- (i) ϕ is (p, q, r) -positive;
- (ii) $\tilde{\phi} : \text{OMAX}^p(\mathcal{S}) \otimes_{\max} \text{OMAX}^q(\mathcal{T}) \rightarrow \mathcal{R}$ is r -positive;
- (iii) $\tilde{\phi} : \text{OMAX}^p(\mathcal{S}) \otimes_{\max} \text{OMAX}^q(\mathcal{T}) \rightarrow \text{OMIN}^r(\mathcal{R})$ is completely positive.

Proof. The equivalence between (ii) and (iii) follows from [29, Theorem 3.7]. Suppose that (ii) holds, and take $x \in M_p(\mathcal{S})^+, y \in M_q(\mathcal{T})^+$ and $\alpha \in M_{r,pq}$. Since

$$\alpha(x \otimes y)\alpha^* \in M_r(\text{OMAX}^p(\mathcal{S}) \otimes_{\max} \text{OMAX}^q(\mathcal{T}))^+,$$

we have

$$\alpha\phi_{p,q}(x, y)\alpha^* = \alpha\tilde{\phi}_{pq}(x \otimes y)\alpha^* = \tilde{\phi}_{pq}(\alpha(x \otimes y)\alpha^*) \in M_r(\mathcal{R})^+,$$

and so, ϕ is (p, q, r) -positive.

For the direction (i) \implies (ii), we take $z \in M_r(\text{OMAX}^p(\mathcal{S}) \otimes_{\max} \text{OMAX}^q(\mathcal{T}))^+$ and arbitrary $\varepsilon_1, \varepsilon_2, \varepsilon_3 > 0$. By (3), we can take $x \in M_m(\text{OMAX}^p(\mathcal{S}))^+, y \in M_n(\text{OMAX}^q(\mathcal{T}))^+$ and $\alpha \in M_{r,mn}$ satisfying the relation

$$z + \varepsilon_1 I_r \otimes 1_{\mathcal{S}} \otimes 1_{\mathcal{T}} = \alpha(x \otimes y)\alpha^*.$$

By (4), we may also find $x_i \in M_p(\mathcal{S})^+, y_j \in M_q(\mathcal{T})^+$ and $\beta \in M_{m,ps}, \gamma \in M_{n,qt}$ satisfying

$$x + \varepsilon_2 I_m \otimes 1_{\mathcal{S}} = \beta \begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_s \end{pmatrix} \beta^*, \quad y + \varepsilon_3 I_n \otimes 1_{\mathcal{T}} = \gamma \begin{pmatrix} y_1 & & \\ & \ddots & \\ & & y_t \end{pmatrix} \gamma^*.$$

Write

$$\alpha(\beta \otimes \gamma) = (\Theta_{(1,1)} \quad \cdots \quad \Theta_{(i,j)} \quad \cdots \quad \Theta_{(s,t)}) \in M_{r,pqst}$$

with $\Theta_{(i,j)} \in M_{r,pq}$. Then, we have the identity

$$\begin{aligned} & \sum_{i=1}^s \sum_{j=1}^t \Theta_{(i,j)} \phi_{p,q}(x_i, y_j) \Theta_{(i,j)}^* \\ &= \alpha(\beta \otimes \gamma) \begin{pmatrix} \phi_{p,q}(x_1, y_1) & & & \\ & \ddots & & \\ & & \phi_{p,q}(x_i, y_j) & \\ & & & \ddots & \\ & & & & \phi_{p,q}(x_s, y_t) \end{pmatrix} (\beta \otimes \gamma)^* \alpha^* \\ &= \alpha\phi_{m,n}(x + \varepsilon_2 I_m \otimes 1_{\mathcal{S}}, y + \varepsilon_3 I_n \otimes 1_{\mathcal{T}})\alpha^* \end{aligned}$$

which belongs to $M_r(\mathcal{R})^+$ by (p, q, r) -positivity of ϕ . Expanding the last term, we have

$$\begin{aligned} & \alpha\phi_{m,n}(x, y)\alpha^* + \varepsilon_3 \alpha\phi_{m,n}(x, I_n \otimes 1_{\mathcal{T}})\alpha^* + \varepsilon_2 \alpha\phi_{m,n}(I_m \otimes 1_{\mathcal{S}}, y)\alpha^* \\ & \quad + \varepsilon_2 \varepsilon_3 \alpha\phi_{m,n}(I_m \otimes 1_{\mathcal{S}}, I_n \otimes 1_{\mathcal{T}})\alpha^* \\ & \leq \tilde{\phi}_r(z) + (\varepsilon_1 \|\tilde{\phi}(1_{\mathcal{S}} \otimes 1_{\mathcal{T}})\| + \varepsilon_3 \|\alpha\phi_{m,n}(x, I_n \otimes 1_{\mathcal{T}})\alpha^*\| \\ & \quad + \varepsilon_2 \|\alpha\phi_{m,n}(I_m \otimes 1_{\mathcal{S}}, y)\alpha^*\| + \varepsilon_2 \varepsilon_3 \|\alpha\phi_{m,n}(I_m \otimes 1_{\mathcal{S}}, I_n \otimes 1_{\mathcal{T}})\alpha^*\|) I_r \otimes 1_{\mathcal{R}} \end{aligned}$$

We note that x, y and α are independent of the choice of ε_2 and ε_3 . Therefore, we can conclude that $\tilde{\phi}_r(z) \in M_r(\mathcal{R})^+$ by the Archimedean property. This proves that the linearization $\tilde{\phi}$ is r -positive. \square

4. SCHMIDT NUMBERS FOR TRI-PARTITE STATES

We recall that the Schmidt rank of a vector $\eta = \sum_{i=1}^n v_i \otimes w_i \in \mathcal{H}_B \otimes \mathcal{H}_C$ is equal to the rank of the associate map $\lambda_\eta : \mathcal{H}_B \rightarrow \mathcal{H}_C$ given by

$$\lambda_\eta(v) = \sum_{i=1}^n \langle \bar{v}_i | v \rangle w_i.$$

In fact, the correspondence $\eta \mapsto \lambda_\eta$ follows from the natural isomorphisms

$$\mathcal{H}_B \otimes \mathcal{H}_C \simeq (\mathcal{H}_B)^* \otimes \mathcal{H}_C \simeq \mathcal{L}(\mathcal{H}_B, \mathcal{H}_C).$$

Here, \mathcal{H}_B is self-dual because it has a canonical basis, and the second isomorphism is the linearization of the bilinear map

$$(f, w) \in (\mathcal{H}_B)^* \times \mathcal{H}_C \mapsto f(\cdot)w \in \mathcal{L}(\mathcal{H}_B, \mathcal{H}_C).$$

Since the map λ_η follows from the above isomorphisms, it is independent of the tensor expression of η .

Applying the above isomorphism twice, we consider the natural isomorphisms

$$\begin{aligned} \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C &\simeq (\mathcal{H}_A)^* \otimes (\mathcal{H}_B)^* \otimes \mathcal{H}_C \\ &\simeq (\mathcal{H}_A)^* \otimes \mathcal{L}(\mathcal{H}_B, \mathcal{H}_C) \simeq \mathcal{L}(\mathcal{H}_A, \mathcal{L}(\mathcal{H}_B, \mathcal{H}_C)), \end{aligned}$$

to get the analogous notion. We write $\xi \in \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$ by

$$\xi = \sum_{i=1}^n u_i \otimes \eta_i \in \mathcal{H}_A \otimes (\mathcal{H}_B \otimes \mathcal{H}_C)$$

with $u_i \in \mathcal{H}_A$ and $\eta_i \in \mathcal{H}_B \otimes \mathcal{H}_C$. Then, the above isomorphisms maps ξ to the linear map $\Lambda_\xi : \mathcal{H}_A \rightarrow \mathcal{L}(\mathcal{H}_B, \mathcal{H}_C)$ given by

$$\Lambda_\xi(u) = \sum_{i=1}^n \langle \bar{u}_i | u \rangle \lambda_{\eta_i}.$$

Now, we consider the following three numbers:

$$\begin{aligned} \alpha_\xi &= \text{rank} \Lambda_\xi, \\ \beta_\xi &= \dim \bigvee \{ \text{supp } T : T \in \text{ran} \Lambda_\xi \} \\ \gamma_\xi &= \dim \bigvee \{ \text{ran } T : T \in \text{ran} \Lambda_\xi \}. \end{aligned}$$

Here, the support of a linear map means the orthogonal complement of its kernel. Since the map Λ_ξ follows from the above isomorphisms, the map Λ_ξ and three numbers $\alpha_\xi, \beta_\xi, \gamma_\xi$ are independent of the tensor expression of ξ .

Definition 4.1. We call the triplet $(\alpha_\xi, \beta_\xi, \gamma_\xi)$ the *Schmidt rank* of the vector $\xi \in \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$ and write $\text{SR}(\xi) = (\alpha_\xi, \beta_\xi, \gamma_\xi)$.

Theorem 4.2. Suppose that $\xi \in \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$ and $1 \leq p \leq a, 1 \leq q \leq b, 1 \leq r \leq c$. Then the following are equivalent:

- (i) $\alpha_\xi \leq p, \beta_\xi \leq q, \gamma_\xi \leq r$;
- (ii) there exist vectors $\{u_i\}_{i=1}^p \subset \mathcal{H}_A, \{v_j\}_{j=1}^q \subset \mathcal{H}_B, \{w_k\}_{k=1}^r \subset \mathcal{H}_C$ and scalars $c_{i,j,k}$ such that

$$\xi = \sum_{i=1}^p \sum_{j=1}^q \sum_{k=1}^r c_{i,j,k} u_i \otimes v_j \otimes w_k;$$

- (iii) there exist orthonormal vectors $\{u_i\}_{i=1}^p \subset \mathcal{H}_A, \{v_j\}_{j=1}^q \subset \mathcal{H}_B, \{w_k\}_{k=1}^r \subset \mathcal{H}_C$ and scalars $c_{i,j,k}$ such that

$$\xi = \sum_{i=1}^p \sum_{j=1}^q \sum_{k=1}^r c_{i,j,k} u_i \otimes v_j \otimes w_k.$$

Proof. (ii) \Rightarrow (i). Suppose that ξ is given as in the statement (ii). Then we have

$$\xi = \sum_{i=1}^p u_i \otimes \left(\sum_{j=1}^q \sum_{k=1}^r c_{i,j,k} v_j \otimes w_k \right) = \sum_{i=1}^p u_i \otimes \left(\sum_{j=1}^q v_j \otimes \left(\sum_{k=1}^r c_{i,j,k} w_k \right) \right).$$

Therefore, we have the following relation

$$(8) \quad \Lambda_\xi(u)(v) = \sum_{i=1}^p \langle \bar{u}_i | u \rangle \lambda_{\sum_{j=1}^q v_j \otimes (\sum_{k=1}^r c_{i,j,k} w_k)}(v) = \sum_{i=1}^p \sum_{j=1}^q \sum_{k=1}^r c_{i,j,k} \langle \bar{u}_i | u \rangle \langle \bar{v}_j | v \rangle w_k,$$

for $u \in \mathcal{H}_A$ and $v \in \mathcal{H}_B$. This relation tells us that $\bigvee \{\text{ran } T : T \in \text{ran } \Lambda_\xi\}$ is a subspace of $\text{span}\{w_k : 1 \leq k \leq r\}$. Therefore, we have $\gamma_\xi \leq r$. For the inequality $\alpha_\xi \leq p$, it suffices to show that $\text{supp } \Lambda_\xi$ is a subspace of $\text{span}\{\bar{u}_i : 1 \leq i \leq p\}$, or equivalently

$$\langle \bar{u}_i | u \rangle = 0, \quad i = 1, 2, \dots, p \implies u \in \text{Ker } \Lambda_\xi,$$

which follows from (8). It remains to show $\beta_\xi \leq q$. If $\langle \bar{v}_j | v \rangle = 0$ for each $j = 1, 2, \dots, q$, then we have $\Lambda_\xi(u)(v) = 0$ by (8). This means that $\{\bar{v}_j : 1 \leq j \leq q\}^\perp$ is the subspace of $\bigcap \{\text{Ker } T : T \in \text{ran } \Lambda_\xi\}$. Considering their orthogonal complements, we see that the space $\bigvee \{\text{supp } T : T \in \text{ran } \Lambda_\xi\}$ is a subspace of $\text{span}\{\bar{v}_j : 1 \leq j \leq q\}$.

(i) \Rightarrow (iii). We write $\alpha_\xi = \alpha, \beta_\xi = \beta, \gamma_\xi = \gamma$. We take an orthonormal basis $\{\bar{u}_i\}_{i=1}^\alpha$ of the support of Λ_ξ , an orthonormal basis $\{\bar{v}_j\}_{j=1}^\beta$ of $\bigvee \{\text{supp } T : T \in \text{ran } \Lambda_\xi\}$ and an orthonormal basis $\{w_k\}_{k=1}^\gamma$ of $\bigvee \{\text{ran } T : T \in \text{ran } \Lambda_\xi\}$. There exist scalars $c_{i,j,k}$ such that $\Lambda_\xi(\bar{u}_i)(\bar{v}_j) = \sum_{k=1}^\gamma c_{i,j,k} w_k$. It suffices to show $\xi = \sum_{i=1}^\alpha \sum_{j=1}^\beta \sum_{k=1}^\gamma c_{i,j,k} u_i \otimes v_j \otimes w_k$. To do this, we show that

$$\Lambda_\xi = \Lambda_{\sum_{i=1}^\alpha \sum_{j=1}^\beta \sum_{k=1}^\gamma c_{i,j,k} u_i \otimes v_j \otimes w_k}.$$

First of all, we have

$$\Lambda_\xi(\bar{u}_{i_0})(v) = \Lambda_\xi(\bar{u}_{i_0}) \left(\sum_{j=1}^\beta \langle \bar{v}_j | v \rangle \bar{v}_j + v' \right) = \sum_{j=1}^\beta \sum_{k=1}^\gamma c_{i_0,j,k} \langle \bar{v}_j | v \rangle w_k$$

for $v \in \mathcal{H}_A$ with $v' \perp \bar{v}_j$. On the other hand, we have

$$\Lambda_{\sum_{i=1}^{\alpha} \sum_{j=1}^{\beta} \sum_{k=1}^{\gamma} c_{i,j,k} u_i \otimes v_j \otimes w_k}(\bar{u}_{i_0})(v) = \sum_{j=1}^{\beta} \sum_{k=1}^{\gamma} c_{i_0,j,k} \lambda_{v_j \otimes w_k}(v) = \sum_{j=1}^{\beta} \sum_{k=1}^{\gamma} c_{i_0,j,k} \langle \bar{v}_j | v \rangle w_k.$$

For all $u \in \mathcal{H}_A$ with $u \perp \bar{u}_i$, we have

$$\Lambda_{\xi}(u) = 0 = \sum_{i=1}^{\alpha} \langle \bar{u}_i | u \rangle \lambda_{\sum_{j=1}^{\beta} \sum_{k=1}^{\gamma} c_{i,j,k} v_j \otimes w_k} = \Lambda_{\sum_{i=1}^{\alpha} \sum_{j=1}^{\beta} \sum_{k=1}^{\gamma} c_{i,j,k} u_i \otimes v_j \otimes w_k}(u).$$

If we put $c_{i,j,k} = 0$ if $\alpha < i \leq p$ or $\beta < j \leq q$ or $\gamma < k \leq r$, we get the expression. \square

For a vector $\xi \in \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$, we write $\text{SR}(\xi) \leq (p, q, r)$ if the conditions in Theorem 4.2 are satisfied. For a permutation σ in $\{A, B, C\}$, we denote by ξ^{σ} the vector in $\mathcal{H}_{\sigma A} \otimes \mathcal{H}_{\sigma B} \otimes \mathcal{H}_{\sigma C}$ obtained by the flip operator under σ . For a given triplet $S = (s_A, s_B, s_C)$ and a permutation σ , we denote by S^{σ} the triplet $(s_{\sigma A}, s_{\sigma B}, s_{\sigma C})$.

Corollary 4.3. $\text{SR}(\xi) = (\alpha, \beta, \gamma)$ if and only if $\text{SR}(\xi^{\sigma}) = (\alpha, \beta, \gamma)^{\sigma}$

Proof. By Theorem 4.2, we see that $\text{SR}(\xi) \leq (p, q, r)$ if and only if $\text{SR}(\xi^{\sigma}) \leq (p, q, r)^{\sigma}$. We apply this for the cases $p = \alpha, \alpha - 1$ and $q = \beta, \beta - 1$ and $r = \gamma, \gamma - 1$ to get the conclusion. \square

In the case of tensor product $\mathcal{H}_A \otimes \mathcal{H}_B$ of two spaces, the Schmidt rank of a vector must fall down in $\{1, 2, \dots, a \wedge b\}$. We show that all the possible combinations of triplets for $\text{SR}(\xi)$ is given by the set

$$\Sigma_{a,b,c} = \{(\alpha, \beta, \gamma) \in \mathbb{N}^3 : \alpha \leq \beta\gamma, \beta \leq \gamma\alpha, \gamma \leq \alpha\beta, 1 \leq \alpha \leq a, 1 \leq \beta \leq b, 1 \leq \gamma \leq c\}.$$

Proposition 4.4. *There exists $\xi \in \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$ such that $\text{SR}(\xi) = (\alpha, \beta, \gamma)$ if and only if $(\alpha, \beta, \gamma) \in \Sigma_{a,b,c}$.*

Proof. We may assume that $\alpha \leq \beta \leq \gamma$ by Corollary 4.3.

(\implies) Let $\xi \in \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$ with $\text{SR}(\xi) = (\alpha, \beta, \gamma)$. We take a basis $\{u_1, \dots, u_{\alpha}\}$ of $\text{supp} \Lambda_{\xi}$. Then, we have

$$\begin{aligned} \gamma &= \dim(\text{ran} \Lambda_{\xi}(u_1) \vee \dots \vee \text{ran} \Lambda_{\xi}(u_{\alpha})) \leq \sum_{i=1}^{\alpha} \dim \text{ran} \Lambda_{\xi}(u_i) \\ &= \sum_{i=1}^{\alpha} \dim \text{supp} \Lambda_{\xi}(u_i) \leq \alpha\beta. \end{aligned}$$

(\impliedby) Let $\gamma = \beta \cdot k + r$ with $0 \leq r < \beta$. Since $\gamma \leq \alpha\beta$, we have $k \leq \alpha$. We take orthonormal sets $\{u_i\}_{i=1}^{\alpha}$ in \mathcal{H}_A , $\{v_j\}_{j=1}^{\beta}$ in \mathcal{H}_B and $\{w_k\}_{k=1}^{\gamma}$ in \mathcal{H}_C . We consider the vector

$$\xi = \sum_{i=1}^k u_i \otimes \left(\sum_{j=1}^{\beta} v_j \otimes w_{(i-1)\beta+j} \right) + u_{k+1} \otimes \left(\sum_{j=1}^r v_j \otimes w_{k\beta+j} \right) + \sum_{i=1}^{\alpha-k-1} u_{k+1+i} \otimes v_i \otimes w_i.$$

in $\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$. The expression in the last term is legitimate because we assume $\alpha \leq \beta \leq \gamma$. If $\gamma = \alpha\beta$, then $k = \alpha$, so we ignore the last two terms. If $(\alpha - 1)\beta \leq \gamma <$

$\alpha\beta$, then $k = \alpha - 1$, so we ignore the last term. The ranges of $\Lambda_\xi(\bar{u}_i)$ for $1 \leq i \leq k + 1$ are orthogonal. The ranges of $\Lambda_\xi(\bar{u}_i)$ for $k + 2 \leq i \leq \alpha$ are orthogonal and their join is a proper subspace of $\text{ran}\Lambda_\xi(\bar{u}_1)$. Hence, we conclude that the set $\{\Lambda_\xi(\bar{u}_i) : 1 \leq i \leq \alpha\}$ is linearly independent, and so $\alpha = \alpha_\xi$.

Since $\text{supp}\Lambda_\xi(\bar{u}_i) = \text{span}\{\bar{v}_j : 1 \leq j \leq \beta\}$ for $1 \leq i \leq k$ and $\text{supp}\Lambda_\xi(\bar{u}_i)$ is a subspace of $\text{span}\{\bar{v}_j : 1 \leq j \leq \beta\}$ for $k + 1 \leq i \leq \alpha$, we have $\beta = \beta_\xi$. Finally, the join of the ranges of $\Lambda_\xi(\bar{u}_i)$ for $1 \leq i \leq \alpha$ is $\text{span}\{w_k : 1 \leq k \leq \gamma\}$, and so $\gamma = \gamma_\xi$. \square

Corollary 4.5. *Let $\xi \in \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$ with $\text{SR}(\xi) = (\alpha, \beta, \gamma)$. If one of α, β, γ is 1, then the other two are equal.*

We recall [10] that the dual cone \mathbb{V}_s of the convex cone of all s -positive linear maps may be described in terms of Schmidt ranks. More precisely, \mathbb{V}_s consists of all bi-partite unnormalized states ϱ which can be expressed by $\varrho = \sum_i |\xi_i\rangle\langle\xi_i|$, where Schmidt rank of ξ_i is less than or equal to s . A bi-partite state ϱ is separable if and only if it belongs to \mathbb{V}_1 by definition. If ϱ belongs to \mathbb{V}_s but does not belong to \mathbb{V}_{s-1} then we say that ϱ has Schmidt number s . This motivates the following:

Definition 4.6. For a tri-partite unnormalized state $\varrho \in M_A \otimes M_B \otimes M_C$, we write $\text{SN}(\varrho) \leq (p, q, r)$ if there exist $\xi_1, \dots, \xi_n \in \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$ such that $\varrho = \sum_{i=1}^n |\xi_i\rangle\langle\xi_i|$ and $\text{SR}(\xi_i) \leq (p, q, r)$ for each $i = 1, 2, \dots, n$. We denote by $\mathbb{S}_{p,q,r}$ the set of all tri-partite unnormalized states ϱ in $M_A \otimes M_B \otimes M_C$ with the property $\text{SN}(\varrho) \leq (p, q, r)$.

The set $\mathbb{S}_{p,q,r}$ is, by definition, the cone generated by the set \mathcal{E} of all rank one projections onto unit vectors ξ with $\text{SR}(\xi) \leq (p, q, r)$. This is equivalent to satisfying the conditions

$$\text{SR}(\xi) \leq (p, b, c), \quad \text{SR}(\xi) \leq (a, q, c), \quad \text{SR}(\xi) \leq (a, b, r),$$

simultaneously. The first condition tells us that $\text{rank}\xi \leq p$ with respect to the A - BC bi-partition of $\xi \in \mathcal{H}_A \otimes (\mathcal{H}_B \otimes \mathcal{H}_C)$, and so the set of all unit vectors ξ with $\text{SR}(\xi) \leq (p, b, c)$ is compact, and same for the other two conditions by Corollary 4.3. Therefore, we see that the set of all unit vectors ξ with $\text{SR}(\xi) \leq (p, q, r)$ is compact. Since the map $|\xi\rangle \mapsto |\xi\rangle\langle\xi|$ is continuous, the set \mathcal{E} is also compact. Hence, we conclude that the set of states in $\mathbb{S}_{p,q,r}$ is also compact, by Caratheodory's theorem [22, Theorem 17.2].

Theorem 4.2 tells us that $\text{SN}(\varrho) \leq (1, 1, 1)$ if and only if ϱ is the convex sum of pure product states, which is nothing but the definition of (genuine) separability. For a state $\varrho \in M_A \otimes M_B \otimes M_C$, it is also clear that $\text{SN}(\varrho) \leq (1, b, c)$ if and only if it is A - BC separable, that is, it is separable as a bi-partite state in $M_A \otimes (M_B \otimes M_C)$. We also see that $\text{SN}(\varrho) \leq (a, 1, c)$ if and only if it is B - CA separable, and $\text{SN}(\varrho) \leq (a, b, 1)$ if and only if it is C - AB separable.

The property $\text{SN}(\xi) \leq (p, q, r)$ can be described in terms of positivity in the maximal tensor product of super maximal operator systems.

Theorem 4.7. *An unnormalized state $\rho \in M_A \otimes M_B \otimes M_C$ belongs to $\mathbb{S}_{p,q,r}$ if and only if we have*

$$\varrho \in [\text{OMAX}^p(M_A) \otimes_{\max} \text{OMAX}^q(M_B) \otimes_{\max} \text{OMAX}^r(M_C)]^+.$$

Proof. (\Leftarrow) Suppose that ϱ belongs to the positive cone in the statement, and take arbitrary $\varepsilon_i > 0$ for $i = 1, 2, 3, 4, 5$. By (3), we can write

$$\varrho + \varepsilon_1 1 = \alpha(X \otimes X')\alpha^*$$

for $X \in M_\ell(\text{OMAX}^p(M_A))^+$, $X' \in M_{\ell'}(\text{OMAX}^q(M_B) \otimes_{\max} M_n(\text{OMAX}^r(M_C)))^+$ and $\alpha \in M_{1,\ell\ell'}$, and

$$X' + \varepsilon_2 I_{\ell'} \otimes 1 = \alpha'(Y \otimes Z)\alpha'^*$$

for $Y \in M_m(\text{OMAX}^q(M_B))^+$, $Z \in M_n(\text{OMAX}^r(M_C))^+$ and $\alpha' \in M_{\ell',mn}$. Moreover, each X, Y, Z can be written as

$$X + \varepsilon_3 I_\ell \otimes 1 = \beta \text{Diag}(X_1, \dots, X_s) \beta^*,$$

$$Y + \varepsilon_4 I_m \otimes 1 = \gamma \text{Diag}(Y_1, \dots, Y_t) \gamma^*,$$

$$Z + \varepsilon_5 I_n \otimes 1 = \delta \text{Diag}(Z_1, \dots, Z_u) \delta^*,$$

for $X_i \in M_p(M_A)^+$, $Y_j \in M_q(M_B)^+$, $Z_k \in M_r(M_C)^+$, and scalar matrices $\beta \in M_{l,ps}$, $\gamma \in M_{m,qt}$ and $\delta \in M_{n,ru}$, by (4). Here, we denote by $\text{Diag}(A_1, \dots, A_n)$ for the $n \times n$ block diagonal matrix with diagonal entries A_1, \dots, A_n .

Combining them and putting $\Theta := \alpha(I_\ell \otimes \alpha')(\beta \otimes \gamma \otimes \delta) \in M_{1,pqrstu}$, we see that the set Ω consisting

$$\Theta (\text{Diag}(X_1, \dots, X_s) \otimes \text{Diag}(Y_1, \dots, Y_t) \otimes \text{Diag}(Z_1, \dots, Z_u)) \Theta^* \in M_A \otimes M_B \otimes M_C$$

through $\Theta \in M_{1,pqrstu}$, $X_i \in M_p(M_A)^+$, $Y_j \in M_q(M_B)^+$, $Z_k \in M_r(M_C)^+$ and $s, t, u = 1, 2, \dots$ is dense in

$$[\text{OMAX}^p(M_A) \otimes_{\max} \text{OMAX}^q(M_B) \otimes_{\max} \text{OMAX}^r(M_C)]^+.$$

Because the set of states in $\mathbb{S}_{p,q,r}$ is compact, it is enough to show that each $\rho \in \Omega$ satisfies $\text{SN}(\rho) \leq (p, q, r)$ by normalization.

We write $\varrho \in \Omega$ as the summation

$$\rho = \sum_{i=1}^s \sum_{j=1}^t \sum_{k=1}^u \Theta \text{Diag}(0, \dots, 0, X_i \otimes Y_j \otimes Z_k, 0, \dots, 0) \Theta^*$$

and consider only (i_0, j_0, k_0) -th term. We let $U_i \in \mathbb{C}^p \otimes \mathcal{H}_A$ be the i -th column of $X_{i_0}^{1/2}$, $V_j \in \mathbb{C}^q \otimes \mathcal{H}_B$ the j -th column of $Y_{j_0}^{1/2}$ and $W_k \in \mathbb{C}^r \otimes \mathcal{H}_C$ the k -th column of $Z_{k_0}^{1/2}$

($1 \leq i \leq pa, 1 \leq j \leq qb, 1 \leq k \leq rc$). Then, we have

$$\begin{aligned}
& \Theta \text{Diag}(0, \dots, 0, X_{i_0} \otimes Y_{j_0} \otimes Z_{k_0}, 0, \dots, 0) \Theta^* \\
&= \sum_{i=1}^{pa} \sum_{j=1}^{qb} \sum_{k=1}^{rc} \Theta \text{Diag}(0, \dots, 0, U_i U_i^* \otimes V_j V_j^* \otimes W_k W_k^*, 0, \dots, 0) \Theta^* \\
&= \sum_{i=1}^{pa} \sum_{j=1}^{qb} \sum_{k=1}^{rc} \left[\Theta \begin{pmatrix} 0 \\ \vdots \\ U_i \otimes V_j \otimes W_k \\ \vdots \\ 0 \end{pmatrix} \right] \left[\Theta \begin{pmatrix} 0 \\ \vdots \\ U_i \otimes V_j \otimes W_k \\ \vdots \\ 0 \end{pmatrix} \right]^*.
\end{aligned}$$

We write $U_i = (u_1, \dots, u_p)^t \in \mathbb{C}^p \otimes \mathcal{H}_A$, $V_j = (v_1, \dots, v_q)^t \in \mathbb{C}^q \otimes \mathcal{H}_B$ and $W_k = (w_1, \dots, w_r)^t \in \mathbb{C}^r \otimes \mathcal{H}_C$. Then, the term in the square bracket is the linear combination of $\{u_i \otimes v_j \otimes w_k : 1 \leq i \leq p, 1 \leq j \leq q, 1 \leq k \leq r\}$. By Theorem 4.2, we see that $\text{SN}(\varrho) \leq (p, q, r)$. The converse is merely the reverse of the above argument. \square

The proof of Theorem 4.7 actually shows that the relation

$$\Omega = [\text{OMAX}^p(M_A) \otimes_{\max} \text{OMAX}^q(M_B) \otimes_{\max} \text{OMAX}^r(M_C)]^+$$

holds.

So far, we have focused on tri-partite states, in order to avoid the excessive notations. Many parts in this section can be extended for multi-partite states. By the isomorphisms

$$\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \dots \otimes \mathcal{H}_{n-1} \otimes \mathcal{H}_n \simeq \mathcal{H}_1^* \otimes \mathcal{H}_2^* \otimes \dots \otimes \mathcal{H}_{n-1}^* \otimes \mathcal{H}_n \simeq \mathcal{L}(\mathcal{H}_1, \mathcal{L}(\mathcal{H}_2, \dots, \mathcal{L}(\mathcal{H}_{n-1}, \mathcal{H}_n)))$$

for finite dimensional Hilbert spaces $\mathcal{H}_i = \mathbb{C}^{d_i}$, we associate each ξ in $\bigotimes_{i=1}^n \mathcal{H}_i$ with the linear map $\Lambda_\xi : \mathcal{H}_1 \rightarrow \mathcal{L}(\mathcal{H}_2, \dots, \mathcal{L}(\mathcal{H}_{n-1}, \mathcal{H}_n))$. Now, we consider the following n (≥ 3) numbers:

$$\begin{aligned}
\alpha_\xi^1 &= \dim \text{supp} \Lambda_\xi \\
\alpha_\xi^2 &= \dim \bigvee \{ \text{supp } T_1 : T_1 \in \text{ran} \Lambda_\xi \} \\
\alpha_\xi^3 &= \dim \bigvee \{ \text{supp } T_2 : T_2 \in \text{ran} T_1, T_1 \in \text{ran} \Lambda_\xi \} \\
&\vdots \\
\alpha_\xi^{n-1} &= \dim \bigvee \{ \text{supp } T_{n-2} : T_{n-2} \in \text{ran} T_{n-3}, \dots, T_2 \in \text{ran} T_1, T_1 \in \text{ran} \Lambda_\xi \} \\
\alpha_\xi^n &= \dim \bigvee \{ \text{ran } T_{n-2} : T_{n-2} \in \text{ran} T_{n-3}, \dots, T_2 \in \text{ran} T_1, T_1 \in \text{ran} \Lambda_\xi \}.
\end{aligned}$$

When $n = 2$, it is natural to define $\alpha_\xi^2 = \dim \text{ran} \Lambda_\xi$ in the above context. We define the Schmidt rank of ξ by the n -tuple $(\alpha_\xi^1, \dots, \alpha_\xi^n)$ and denote it by $SR(\xi)$. We have

$$\alpha_\xi^1 \leq s_1, \dots, \alpha_\xi^n \leq s_n$$

if and only if there exist vectors $\{u_{\iota_1}^1\}_{\iota_1=1}^{s_1} \subset \mathcal{H}_1, \dots, \{u_{\iota_n}^n\}_{\iota_n=1}^{s_n} \subset \mathcal{H}_n$ and scalars $c_{\iota_1, \dots, \iota_n}$ such that

$$\xi = \sum_{\iota_n=1}^{s_n} \cdots \sum_{\iota_1=1}^{s_1} c_{\iota_1, \dots, \iota_n} u_{\iota_1}^1 \otimes \cdots \otimes u_{\iota_n}^n.$$

In this case, we write $SR(\xi) \leq (s_1, \dots, s_n)$.

For a multi-partite unnormalized state $\varrho \in \bigotimes_{i=1}^n \mathcal{L}(\mathcal{H}_i)$, we write $SN(\varrho) \leq (s_1, \dots, s_n)$ if there exist $\xi_1, \dots, \xi_m \in \bigotimes_{i=1}^n \mathcal{H}_i$ such that $\varrho = \sum_{i=1}^m |\xi_i\rangle\langle\xi_i|$ and $SR(\xi_i) \leq (s_1, \dots, s_n)$ for each $i = 1, 2, \dots, m$. We denote by $\mathbb{S}_{s_1, \dots, s_n}$ the set of all multi-partite unnormalized states ϱ in $\bigotimes_{i=1}^n \mathcal{L}(\mathcal{H}_i)$ with the property $SN(\varrho) \leq (s_1, \dots, s_n)$. Then, we have

$$\mathbb{S}_{s_1, \dots, s_n} = [\text{OMAX}^{s_1}(\mathcal{L}(\mathcal{H}_1)) \otimes_{\max} \cdots \otimes_{\max} \text{OMAX}^{s_n}(\mathcal{L}(\mathcal{H}_n))]^{+}.$$

In the bi-partite case, the Schumidt number of $\varrho \in M_m \otimes M_n$ is less than or equal to k in the usual sense if and only if $SN(\varrho) \leq (k, k)$ if and only if $SN(\varrho) \leq (m, k)$ because $SR(\xi) = (j, k)$ cannot occur for $j \neq k$ due to the first isomorphism theorem. By [29, Theorem 4.6], we have $M_m = \text{OMAX}^m(M_m)$. Therefore, the Schmidt number of ϱ is less than or equal to k in the usual sense if and only if

$$\varrho \in [\text{OMAX}^m(M_m) \otimes_{\max} \text{OMAX}^k(M_n)]^{+} = M_m(\text{OMAX}^k(M_n))^{+},$$

which was proved in [16, Theorem 5].

5. DUALITY

For $\varrho \in M_A \otimes M_B \otimes M_C$ and a bi-linear map $\phi : M_A \times M_B \rightarrow M_C$, the bi-linear pairing $\langle \varrho, \phi \rangle$ is defined [19] by

$$(9) \quad \langle \varrho, \phi \rangle = \langle \varrho, C_\phi \rangle = \text{Tr}(C_\phi \varrho^t).$$

If $\varrho = u \otimes v \otimes w \in M_A \otimes M_B \otimes M_C$, then the pairing is given by

$$\langle u \otimes v \otimes w, \phi \rangle = \langle \phi(u, v), w \rangle = \text{Tr}(\phi(u, v)w^t).$$

We denote by $\mathbb{P}_{p,q,r}$ the convex cone consisting of all (p, q, r) -positive bi-linear maps from $M_A \times M_B$ to M_C .

By (5) and (6), we have the complete order isomorphisms

$$\begin{aligned} & (\text{OMAX}^p(M_A) \otimes_{\max} \text{OMAX}^q(M_B) \otimes_{\max} \text{OMAX}^r(M_C))^{*} \\ & \simeq \mathcal{L}(\text{OMAX}^p(M_A) \otimes_{\max} \text{OMAX}^q(M_B), \text{OMAX}^r(M_C)^{*}) \\ & \simeq \mathcal{L}(\text{OMAX}^p(M_A) \otimes_{\max} \text{OMAX}^q(M_B), \text{OMIN}^r(M_C)). \end{aligned}$$

In these isomorphisms, a functional φ on $\text{OMAX}^p(M_A) \otimes_{\max} \text{OMAX}^q(M_B) \otimes_{\max} \text{OMAX}^r(M_C)$ is positive if and only if its associated linear map

$$\tilde{\phi} : \text{OMAX}^p(M_A) \otimes_{\max} \text{OMAX}^q(M_B) \rightarrow \text{OMIN}^r(M_C)$$

is completely positive. By Theorem 3.5 and Theorem 4.7, we have the following duality between the convex cones $\mathbb{S}_{p,q,r}$ and $\mathbb{P}_{p,q,r}$. We give here a direct elementary proof.

Theorem 5.1. *The convex cones $\mathbb{S}_{p,q,r}$ and $\mathbb{P}_{p,q,r}$ are dual to each other. In other words, $\varrho \in \mathbb{S}_{p,q,r}$ if and only if $\langle \varrho, \phi \rangle \geq 0$ for each $\phi \in \mathbb{P}_{p,q,r}$.*

Proof. By the separation theorem for a point outside of a closed convex set, it is sufficient to show that a bilinear map $\phi : M_A \otimes M_B \rightarrow M_C$ is (p, q, r) -positive if and only if $\langle \phi, \varrho \rangle \geq 0$ for all $\varrho \in \mathbb{S}_{p,q,r}$. Suppose that $\phi : M_A \otimes M_B \rightarrow M_C$ is (p, q, r) -positive and $\xi \in \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$ with $\text{SR}(\xi) \leq (p, q, r)$. By Theorem 4.2, there exist vectors $\{u_i\}_{i=1}^p \subset \mathcal{H}_A$, $\{v_j\}_{j=1}^q \subset \mathcal{H}_B$, $\{w_k\}_{k=1}^r \subset \mathcal{H}_C$ and scalars $c_{i,j,k}$ such that

$$\xi = \sum_{i=1}^p \sum_{j=1}^q \sum_{k=1}^r c_{i,j,k} u_i \otimes v_j \otimes w_k.$$

Then we have

$$\begin{aligned} \langle \phi, \xi \xi^* \rangle &= \left\langle \phi, \sum_{i,\ell=1}^p \sum_{j,m=1}^q \sum_{k,n=1}^r c_{i,j,k} \bar{c}_{\ell,m,n} u_i u_\ell^* \otimes v_j v_m^* \otimes w_k w_n^* \right\rangle \\ &= \sum_{i,\ell=1}^p \sum_{j,m=1}^q \sum_{k,n=1}^r c_{i,j,k} \bar{c}_{\ell,m,n} \langle \tilde{\phi}(u_i u_\ell^* \otimes v_j v_m^*), w_k w_n^* \rangle_{M_C} \\ &= \left\langle \left[\sum_{i,\ell=1}^p \sum_{j,m=1}^q c_{i,j,k} \bar{c}_{\ell,m,n} \tilde{\phi}(u_i u_\ell^* \otimes v_j v_m^*) \right]_{k,n}, [w_k w_n^*]_{k,n} \right\rangle_{M_r(M_C)}. \end{aligned}$$

If we denote by α the $r \times pq$ matrix whose entries are given by $\alpha_{k,(i,j)} = c_{i,j,k}$, then the above quantity coincides with the following:

$$\left\langle \alpha \phi_{p,q} \left(\begin{pmatrix} u_1 \\ \vdots \\ u_p \end{pmatrix} (u_1^* \cdots u_p^*), \begin{pmatrix} v_1 \\ \vdots \\ v_q \end{pmatrix} (v_1^* \cdots v_q^*) \right) \alpha^*, \begin{pmatrix} w_1 \\ \vdots \\ w_r \end{pmatrix} (w_1^* \cdots w_r^*) \right\rangle_{M_r(M_C)},$$

which is nonnegative, because ϕ is (p, q, r) -positive. Since every positive matrix is the sum of rank one positive matrices, we get the converse by reversing of the above argument. \square

Recall that a Hermitian matrix $W \in M_A \otimes M_B$ is said to be s -block positive if $\langle W, \varrho \rangle \geq 0$ for each ϱ whose Schmidt number is less than or equal to s . In this context, it is reasonable to say that a Hermitian matrix $W \in M_A \otimes M_B \otimes M_C$ is (p, q, r) -block positive if $\langle W, \varrho \rangle \geq 0$ for all $\varrho \in \mathbb{S}_{p,q,r}$. Then Theorem 5.1 and (6) together with [11, Proposition 1.16] tells us the following.

Corollary 5.2. *Suppose that $\phi : M_A \times M_B \rightarrow M_C$ is a bi-linear map. Then the following are equivalent:*

- (i) ϕ is (p, q, r) -positive;
- (ii) the Choi matrix $C_\phi \in M_A \otimes M_B \otimes M_C$ is (p, q, r) -block positive;
- (iii) C_ϕ belongs to $(\text{OMIN}^p(M_A) \otimes_{\min} \text{OMIN}^q(M_B) \otimes_{\min} \text{OMIN}^r(M_C))^+$.

A tri-partite state ϱ is called bi-separable if it is in the convex hull of all A - BC , B - CA and C - AB separable states. This happens if and only if ϱ belongs to the convex

hull of the convex sets $\mathbb{S}_{1,b,c}$, $\mathbb{S}_{a,1,c}$ and $\mathbb{S}_{a,b,1}$. The dual of this convex hull is the intersection of dual cones. Therefore, we have the the following. Recall that ϱ is genuinely entangled if it is not bi-separable.

Corollary 5.3. *A state $\varrho \in M_A \otimes M_B \otimes M_C$ is genuinely entangled if and only if there exists $\phi \in \mathbb{P}_{1,b,c} \cap \mathbb{P}_{a,1,c} \cap \mathbb{P}_{a,b,1}$ satisfying $\langle \varrho, \phi \rangle < 0$.*

Therefore, an $abc \times abc$ self-adjoint matrix W is a witness for genuine entanglement if and only if it is the Choi matrix of a bi-linear map which belongs to $\mathbb{P}_{1,b,c} \cap \mathbb{P}_{a,1,c} \cap \mathbb{P}_{a,b,1}$. In the next section, we construct such examples for three qubit case of $a = b = c = 2$.

On the other hand, a tri-partite state ϱ is said to be fully bi-separable if it is bi-separable with respect to any possible bi-partitions. This is the case if and only if it is in the intersection of $\mathbb{S}_{1,b,c}$, $\mathbb{S}_{a,1,c}$ and $\mathbb{S}_{a,b,1}$. See [19, 27] for examples of three qubit fully bi-separable states which are not genuinely separable.

Recall that the notion of s -positivity of a linear map is invariant under taking the dual map. We proceed to consider what happens for (p, q, r) -positivity of bi-linear maps. For a permutation σ in the set $\{A, B, C\}$, we define the bi-linear map

$$\phi^\sigma : M_{\sigma A} \times M_{\sigma B} \rightarrow M_{\sigma C}$$

by

$$\langle \phi^\sigma(x_{\sigma A}, x_{\sigma B}), x_{\sigma C} \rangle = \langle \phi(x_A, x_B), x_C \rangle, \quad x_A \in M_A, x_B \in M_B, x_C \in M_C.$$

The isomorphism from $M_A \otimes M_B \otimes M_C$ onto $M_{\sigma A} \otimes M_{\sigma B} \otimes M_{\sigma C}$ given by the flip map under σ will be denote by $\varrho \mapsto \varrho^\sigma$. By Corollary 4.3, we have

$$\text{SN}(\varrho) \leq (p, q, r) \iff \text{SN}(\varrho^\sigma) \leq (p, q, r)^\sigma.$$

Since $\langle \phi, \varrho \rangle = \langle \phi^\sigma, \varrho^\sigma \rangle$, the following is immediate by Theorem 5.1.

Corollary 5.4. *Suppose that $\varrho \in M_A \otimes M_B \otimes M_C$ and $\phi : M_A \times M_B \rightarrow M_C$ is a bi-linear map. Then, ϕ is (p, q, r) -positive if and only if ϕ^σ is $(p, q, r)^\sigma$ -positive.*

By Proposition 4.4 and Corollary 4.5, we also have the following:

- $\text{SN}(\varrho) \leq (p, q, r)$ and $r \geq pq$, then $\text{SN}(\varrho) \leq (p, q, pq)$.
- If $\text{SN}(\varrho) \leq (1, q, r)$, then $\text{SN}(\varrho) \leq (1, q \wedge r, q \wedge r)$.

This reflects Proposition 3.1 and Proposition 3.2, respectively, by duality. Applying Corollary 5.4, it is clear that the role of (p, q, r) may be permuted together with (A, B, C) in Propositions 3.1 and 3.2 if domains and ranges are matrix algebras.

Corollary 5.5. *Suppose that $\phi : M_A \times M_B \rightarrow M_C$ is a bi-linear map. For $q, r \in \mathbb{N}$, the following are equivalent:*

- (i) ϕ is (p, q, r) -positive for all $p \in \mathbb{N}$;
- (ii) ϕ is (p, q, r) -positive for some $p \geq qr$;
- (iii) ϕ is (qr, q, r) -positive.

Moreover, similar equivalence relations hold for fixed $p, r \in \mathbb{N}$.

Corollary 5.6. *Suppose that $\phi : M_A \times M_B \rightarrow M_C$ is a bi-linear map. Then, ϕ is $(p, q, 1)$ -positive if and only if ϕ is $(p \wedge q, p \wedge q, 1)$ -positive.*

The following tells us that the inclusion relations between cones $\mathbb{P}_{p,q,r}$ are given by product order of triplets (p, q, r) , and they are distinct for different triplets in $\Sigma_{a,b,c}$.

Proposition 5.7. *For (p_1, q_1, r_1) and (p_2, q_2, r_2) in $\Sigma_{a,b,c}$, the following are equivalent:*

- (i) $\mathbb{S}_{p_1, q_1, r_1} \subset \mathbb{S}_{p_2, q_2, r_2}$;
- (ii) $\mathbb{P}_{p_1, q_1, r_1} \supset \mathbb{P}_{p_2, q_2, r_2}$;
- (iii) $p_1 \leq p_2, q_1 \leq q_2$ and $r_1 \leq r_2$.

In particular, we have $\mathbb{S}_{p_1, q_1, r_1} = \mathbb{S}_{p_2, q_2, r_2}$ if and only if $\mathbb{P}_{p_1, q_1, r_1} = \mathbb{P}_{p_2, q_2, r_2}$ if and only if $(p_1, q_1, r_1) = (p_2, q_2, r_2)$.

Proof. The equivalence (i) \iff (ii) and the implication (iii) \implies (i) follow from Theorem 5.1 and Theorem 4.2, respectively.

For any $(p_1, q_1, r_1) \in \Sigma_{a,b,c}$, we have constructed in Proposition 4.4 a vector $\xi \in \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$ with $\text{SR}(\xi) = (p_1, q_1, r_1)$, and so $|\xi\rangle\langle\xi| \in \mathbb{S}_{p_1, q_1, r_1}$. If $|\xi\rangle\langle\xi| = \sum_k |\xi_k\rangle\langle\xi_k|$ then each $|\xi_k\rangle$ is a scalar multiplication of $|\xi\rangle$. Therefore, if $p_2 < p_1$ then $|\xi\rangle\langle\xi|$ never belongs to $\mathbb{S}_{p_2, q_2, r_2}$, and so $\mathbb{S}_{p_1, q_1, r_1} \subset \mathbb{S}_{p_2, q_2, r_2}$ does not hold. The same is true when $q_2 < q_1$ or $r_2 < r_1$. This completes the proof of the implication (i) \implies (iii). \square

In order to construct bi-linear maps, we need to identify them by various objects like functionals, matrices and linear maps. For this purpose, we consider the following algebraic isomorphisms:

$$\begin{array}{ccccc} \mathcal{BL}(M_A, M_B; M_C) & \xrightarrow{\simeq} & \mathcal{L}(M_A \otimes M_B, M_C) & \xrightarrow{\simeq} & (M_{ABC})^* \xrightarrow{\simeq} M_{ABC} \\ & & \downarrow \simeq & & \downarrow \simeq \\ & & \mathcal{L}(M_{AB}, M_C) & & \mathcal{L}(M_A, M_{BC}) \end{array}$$

We denote by

$$\begin{aligned} A_\phi &\in \mathcal{L}(M_A \otimes M_B, M_C), & B_\phi &\in (M_{ABC})^*, & C_\phi &\in M_{ABC}, \\ D_\phi &\in \mathcal{L}(M_{AB}, M_C), & E_\phi &\in \mathcal{L}(M_A, M_{BC}) \end{aligned}$$

elements associated with a bilinear map $\phi : M_A \otimes M_B \rightarrow M_C$ by the above isomorphisms, respectively. It is worth to note that C_ϕ is the Choi matrix of ϕ . The properties of A_ϕ , B_ϕ and C_ϕ corresponding to (p, q, r) -positivity of ϕ are characterized in Theorem 3.5, Theorem 5.1 and Corollary 5.2, respectively.

Theorem 5.8. *Suppose that $\phi : M_A \times M_B \rightarrow M_C$ is a bi-linear map. The following are equivalent:*

- (i) ϕ is (p, q, r) -positive;
- (ii) $(D_\phi)_r : M_r(M_{AB}) \rightarrow M_r(M_C)$ maps $\mathbb{S}_{r,p,q}$ into $M_r(M_C)^+$;

- (iii) $(E_\phi)_p : M_p(M_A) \rightarrow M_p(M_{BC})$ maps positive matrices to (p, q, r) -block positive matrices.

Proof. (i) \iff (ii). By Theorem 3.5, $\phi : M_A \times M_B \rightarrow M_C$ is (p, q, r) -positive if and only if the map

$$(A_\phi)_r : M_r(\text{OMAX}^p(M_A) \otimes_{\max} \text{OMAX}^q(M_B)) \rightarrow M_r(M_C)$$

is positive. On the other hand, we have

$$\begin{aligned} \mathbb{S}_{r,p,q} &= (\text{OMAX}^r(M_r) \otimes_{\max} \text{OMAX}^p(M_A) \otimes_{\max} \text{OMAX}^q(M_B))^+ \\ &= M_r(\text{OMAX}^p(M_A) \otimes_{\max} \text{OMAX}^q(M_B))^+. \end{aligned}$$

by Theorem 4.7.

- (i) \iff (iii). By (5), we have a complete order isomorphism

$$\begin{aligned} &(\text{OMAX}^p(M_A) \otimes_{\max} \text{OMAX}^q(M_B) \otimes_{\max} \text{OMAX}^r(M_C))^* \\ &\simeq \mathcal{L}(\text{OMAX}^p(M_A), (\text{OMAX}^q(M_B) \otimes_{\max} \text{OMAX}^r(M_C))^*). \end{aligned}$$

Thus, $\phi : M_A \times M_B \rightarrow M_C$ is (p, q, r) -positive if and only if B_ϕ is positive in the left side if and only if E_ϕ is completely positive in the right side if and only if

$$(E_\phi)_p : M_p(M_A) \rightarrow M_p((\text{OMAX}^q(M_B) \otimes_{\max} \text{OMAX}^r(M_C))^*)$$

is positive. By [29, Theorem 3.7], we have $M_p = \text{OMIN}^p(M_p)$. Therefore, we have the complete order isomorphism

$$\begin{aligned} &M_p((\text{OMAX}^q(M_B) \otimes_{\max} \text{OMAX}^r(M_C))^*) \\ &\simeq \text{OMIN}^p(M_p) \otimes_{\min} \text{OMIN}^q(M_B) \otimes_{\min} \text{OMIN}^r(M_C) \end{aligned}$$

by (6) and [11, Proposition 1.16]. The conclusion follows from Corollary 5.2. \square

Corollary 5.9. *For a bi-linear map $\phi : M_A \times M_B \rightarrow M_C$, we have the following:*

- (i) ϕ is (a, b, r) -positive if and only if D_ϕ is r -positive.
- (ii) ϕ is $(p, q, 1)$ -positive if and only if D_ϕ maps $\mathbb{V}_{p \wedge q}$ into M_C^+ .
- (iii) ϕ is (p, b, c) -positive if and only if E_ϕ is p -positive.
- (iv) ϕ is $(1, q, r)$ -positive if and only if E_ϕ maps M_A^+ to $q \wedge r$ -block positive matrices.

If we express (p, q, r) -positive bi-linear maps by their Choi matrices, then they are witnesses for entanglement ϱ which does not satisfy $\text{SN}(\varrho) \leq (p, q, r)$. Therefore, it is very useful to know how the Choi matrix is changed when we take the dual ϕ^σ with respect to the permutation σ . For this purpose, we see easily that the following diagram commutes:

$$(10) \quad \begin{array}{ccc} \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C & \xrightarrow{C_\phi} & \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C \\ \downarrow & & \downarrow \\ \mathcal{H}_{\sigma A} \otimes \mathcal{H}_{\sigma B} \otimes \mathcal{H}_{\sigma C} & \xrightarrow{C_{\phi^\sigma}} & \mathcal{H}_{\sigma A} \otimes \mathcal{H}_{\sigma B} \otimes \mathcal{H}_{\sigma C} \end{array}$$

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where the vertical arrows are the flip operator under the permutation σ . Therefore, taking dual with respect to a permutation corresponds to changing the order of columns and rows in the Choi matrices.

6. EXAMPLES

In this section, we exhibit examples of (p, q, r) -positive bi-linear maps between 2×2 matrices. When we write down Choi matrices, we always use the lexicographic order:

$$(11) \quad |000\rangle, \quad |001\rangle, \quad |010\rangle, \quad |011\rangle, \quad |100\rangle, \quad |101\rangle, \quad |110\rangle, \quad |111\rangle,$$

for a basis of $\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C = \mathbb{C}^8$. Motivated by examples in [19], we consider the following 8×8 matrix

$$(12) \quad \begin{pmatrix} s_1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & u_1 \\ \cdot & s_2 & \cdot & \cdot & \cdot & \cdot & u_2 & \cdot \\ \cdot & \cdot & s_3 & \cdot & \cdot & u_3 & \cdot & \cdot \\ \cdot & \cdot & \cdot & s_4 & u_4 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \bar{u}_4 & t_4 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \bar{u}_3 & \cdot & \cdot & t_3 & \cdot & \cdot \\ \cdot & \bar{u}_2 & \cdot & \cdot & \cdot & \cdot & t_2 & \cdot \\ \bar{u}_1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & t_1 \end{pmatrix},$$

with nonnegative numbers s_i, t_i and complex numbers u_i , for $i = 1, 2, 3, 4$, where \cdot denotes zero. This is the Choi matrix of the bi-linear map $\phi : M_2 \times M_2 \rightarrow M_2$ which sends the pair $([x_{ij}], [y_{k\ell}]) \in M_2 \times M_2$ to

$$\begin{pmatrix} s_1 x_{11} y_{11} + s_3 x_{11} y_{22} + t_4 x_{22} y_{11} + t_2 x_{22} y_{22} & u_1 x_{12} y_{12} + u_3 x_{12} y_{21} + \bar{u}_4 x_{21} y_{12} + \bar{u}_2 x_{21} y_{21} \\ u_2 x_{12} y_{12} + u_4 x_{12} y_{21} + \bar{u}_3 x_{21} y_{12} + \bar{u}_1 x_{21} y_{21} & s_2 x_{11} y_{11} + s_4 x_{11} y_{22} + t_3 x_{22} y_{11} + t_1 x_{22} y_{22} \end{pmatrix}.$$

By Proposition 4.4 and duality, all possible kinds of positivity of ϕ are given by

$$(2, 2, 2), \quad (1, 2, 2), \quad (2, 1, 2), \quad (2, 2, 1), \quad (1, 1, 1).$$

By Theorem 3.4, we see that ϕ is $(2, 2, 2)$ -positive if and only if its Choi matrix is positive if and only if

$$\sqrt{s_i t_i} \geq |u_i|$$

for each $i = 1, 2, 3, 4$.

Lemma 6.1. *Let $a, c, b, d \geq 0$ and $\omega, z \in \mathbb{C}$. The inequality*

$$\sqrt{ab} + \sqrt{cd} \geq |\omega| + |z|,$$

holds if and only if the matrix

$$\begin{pmatrix} a + d|\alpha|^2 & \omega \bar{\alpha} + \bar{z} \alpha \\ \bar{\omega} \alpha + z \bar{\alpha} & c + b|\alpha|^2 \end{pmatrix}$$

is positive for all $\alpha \in \mathbb{C}$.

Proof. (\implies) By the Cauchy-Schwartz inequality, we have

$$\begin{aligned}
|\omega\bar{\alpha} + \bar{z}\alpha| &\leq |\omega||\bar{\alpha}| + |\bar{z}||\alpha| \\
&\leq (\sqrt{ab} + \sqrt{cd})|\alpha| \\
&= \sqrt{a}(\sqrt{b}|\alpha|) + (\sqrt{d}|\alpha|)\sqrt{c} \\
&\leq \sqrt{a+d|\alpha|^2}\sqrt{c+b|\alpha|^2}.
\end{aligned}$$

(\impliedby) We first consider the case when $bd \neq 0$. Let

$$\omega z = |\omega z|e^{i\theta} \quad \text{and} \quad \alpha_0 := \left(\frac{ac}{bd}\right)^{\frac{1}{4}} e^{i\theta/2}.$$

Then we have

$$\begin{aligned}
\sqrt{a+d|\alpha_0|^2}\sqrt{c+b|\alpha_0|^2} &= \sqrt{a + \frac{d\sqrt{ac}}{\sqrt{bd}}} \sqrt{c + \frac{b\sqrt{ac}}{\sqrt{bd}}} \\
&= \left(\sqrt{\frac{a}{b}}(\sqrt{ab} + \sqrt{cd}) \sqrt{\frac{c}{d}}(\sqrt{ab} + \sqrt{cd}) \right)^{\frac{1}{2}} \\
&= |\alpha_0|(\sqrt{ab} + \sqrt{cd}).
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
|\omega\bar{\alpha}_0 + \bar{z}\alpha_0| &= \sqrt{|\omega\bar{\alpha}_0 + \bar{z}\alpha_0|^2} \\
&= (|\omega|^2|\alpha_0|^2 + \omega z \bar{\alpha}_0^2 + \bar{\omega} \bar{z} \alpha_0^2 + |z|^2|\alpha_0|^2)^{\frac{1}{2}} \\
&= (|\omega|^2|\alpha_0|^2 + 2|\omega z||\alpha_0|^2 + |z|^2|\alpha_0|^2)^{\frac{1}{2}} = |\alpha_0|(|\omega| + |z|).
\end{aligned}$$

Hence, the inequality

$$\sqrt{ab} + \sqrt{cd} \geq |\omega| + |z|,$$

holds if $bd \neq 0$.

When $bd = 0$, we take $\varepsilon > 0$ and apply the above to the positive matrix

$$\begin{pmatrix} a + (d + \varepsilon)|\alpha|^2 & \omega\bar{\alpha} + \bar{z}\alpha \\ \bar{\omega}\alpha + z\bar{\alpha} & c + (b + \varepsilon)|\alpha|^2 \end{pmatrix},$$

which yields

$$\sqrt{a(b + \varepsilon)} + \sqrt{c(d + \varepsilon)} \geq |\omega| + |z|$$

for any $\varepsilon > 0$. \square

Theorem 6.2. Suppose that $\phi : M_2 \times M_2 \rightarrow M_2$ is a bilinear map given by its Choi matrix as (12), and consider the inequalities

$$(13) \quad \sqrt{s_i t_i} + \sqrt{s_j t_j} \geq |u_i| + |u_j|.$$

Then we have the following:

- (i) ϕ is $(1, 2, 2)$ -positive if and only if (13) hold for $(i, j) = (1, 4)$ and $(2, 3)$.
- (ii) ϕ is $(2, 1, 2)$ -positive if and only if (13) hold for $(i, j) = (1, 3)$ and $(2, 4)$.
- (iii) ϕ is $(2, 2, 1)$ -positive if and only if (13) hold for $(i, j) = (1, 2)$ and $(3, 4)$.

Proof. We will consider the linear map $E_\phi : M_2 \rightarrow M_4$ in Theorem 5.8. The map E_ϕ sends a rank one positive matrix $P_\alpha = \begin{pmatrix} 1 & \bar{\alpha} \\ \alpha & |\alpha|^2 \end{pmatrix}$ to

$$(14) \quad \begin{pmatrix} s_1 + t_4|\alpha|^2 & \cdot & \cdot & u_1\bar{\alpha} + \bar{u}_4\alpha \\ \cdot & s_2 + t_3|\alpha|^2 & u_2\bar{\alpha} + \bar{u}_3\alpha & \cdot \\ \cdot & u_3\bar{\alpha} + \bar{u}_2\alpha & s_3 + t_2|\alpha|^2 & \cdot \\ u_4\bar{\alpha} + \bar{u}_1\alpha & \cdot & \cdot & s_4 + t_1|\alpha|^2 \end{pmatrix}.$$

We see that ϕ is $(1, 2, 2)$ -positive if and only if E_ϕ is positive by Corollary 5.9 (iii), if and only if the above matrix is positive for each complex number α . From this, we get the statement (i) by Lemma 6.1.

For the statement (ii), we consider the permutation σ which sends (A, B, C) to (B, C, A) . Then ϕ is $(2, 1, 2)$ -positive if and only if ϕ^σ is $(1, 2, 2)$ -positive. The basis (11) is changed to

$$|000\rangle, |010\rangle, |100\rangle, |110\rangle, |001\rangle, |011\rangle, |101\rangle, |111\rangle,$$

by the flip operation under σ . Therefore, we have $C_{\phi^\sigma} = UC_\phi U^*$ with the 8×8 unitary U which sends a standard ordered basis to an ordered basis $\{e_1, e_3, e_5, e_7, e_2, e_4, e_6, e_8\}$. Therefore, we have

$$(15) \quad C_{\phi^\sigma} = \begin{pmatrix} s_1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & u_1 \\ \cdot & t_4 & \cdot & \cdot & \cdot & \cdot & \bar{u}_4 & \cdot \\ \cdot & \cdot & s_2 & \cdot & \cdot & u_2 & \cdot & \cdot \\ \cdot & \cdot & \cdot & t_3 & \bar{u}_3 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & u_3 & s_3 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \bar{u}_2 & \cdot & \cdot & t_2 & \cdot & \cdot \\ \cdot & u_4 & \cdot & \cdot & \cdot & \cdot & s_4 & \cdot \\ \bar{u}_1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & t_1 \end{pmatrix},$$

and the result follows from (i).

If we consider the permutation σ which send (A, B, C) to (C, A, B) then the standard ordered basis goes to an ordered basis $\{e_1, e_5, e_2, e_6, e_3, e_7, e_4, e_8\}$, and we have

$$(16) \quad C_{\phi^\sigma} = \begin{pmatrix} s_1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & u_1 \\ \cdot & s_3 & \cdot & \cdot & \cdot & \cdot & u_3 & \cdot \\ \cdot & \cdot & t_4 & \cdot & \cdot & \bar{u}_4 & \cdot & \cdot \\ \cdot & \cdot & \cdot & t_2 & \bar{u}_2 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & u_2 & s_2 & \cdot & \cdot & \cdot \\ \cdot & \cdot & u_4 & \cdot & \cdot & s_4 & \cdot & \cdot \\ \cdot & \bar{u}_3 & \cdot & \cdot & \cdot & \cdot & t_3 & \cdot \\ \bar{u}_1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & t_1 \end{pmatrix}.$$

Therefore, (iii) comes from (i) again. \square

As for the $(1, 1, 1)$ -positivity, we use E_ϕ in Proposition 5.8 together with Lemma 6.1, to get the following:

Theorem 6.3. *Suppose that $\phi : M_2 \times M_2 \rightarrow M_2$ is a bilinear map given by its Choi matrix as (12). Then, ϕ is $(1, 1, 1)$ -positive if and only if the inequality*

$$\begin{aligned} & \sqrt{(s_1 + t_4|\alpha|^2)(s_4 + t_1|\alpha|^2)} + \sqrt{(s_2 + t_3|\alpha|^2)(s_3 + t_2|\alpha|^2)} \\ & \geq |u_1\bar{\alpha} + \bar{u}_4\alpha| + |u_2\bar{\alpha} + \bar{u}_3\alpha| \end{aligned}$$

holds for each $\alpha \in \mathbb{C}$. In particular, this holds when

$$\sum_{i=1}^4 \sqrt{s_i t_i} \geq \sum_{i=1}^4 |u_i|.$$

Proof. The linear map given with the Choi matrix (14) sends a rank one positive matrix $P_\beta = \begin{pmatrix} 1 & \bar{\beta} \\ \beta & |\beta|^2 \end{pmatrix}$ to

$$\begin{pmatrix} (s_1 + t_4|\alpha|^2) + (s_3 + t_2|\alpha|^2)|\beta|^2 & (u_1\bar{\alpha} + \bar{u}_4\alpha)\bar{\beta} + (u_3\bar{\alpha} + \bar{u}_2\alpha)\beta \\ (u_4\bar{\alpha} + \bar{u}_1\alpha)\beta + (u_2\bar{\alpha} + \bar{u}_3\alpha)\bar{\beta} & (s_2 + t_3|\alpha|^2) + (s_4 + t_1|\alpha|^2)|\beta|^2 \end{pmatrix}.$$

The bilinear map ϕ is $(1, 1, 1)$ -positive if and only if the matrix (14) is block positive for all $\alpha \in \mathbb{C}$ by Corollary 5.9 (iv), if and only if the above matrix is positive for all $\alpha, \beta \in \mathbb{C}$. It is equivalent to the inequality by Lemma 6.1.

For the last statement, we note

$$\begin{aligned} |u_1\bar{\alpha} + \bar{u}_4\alpha| + |u_2\bar{\alpha} + \bar{u}_3\alpha| & \leq \left(\sum_{i=1}^4 |u_i| \right) |\alpha| \leq \left(\sum_{i=1}^4 \sqrt{s_i t_i} \right) |\alpha| \\ & = \sqrt{s_1}(\sqrt{t_1}|\alpha|) + (\sqrt{t_4}|\alpha|)\sqrt{s_4} + \sqrt{s_2}(\sqrt{t_2}|\alpha|) + (\sqrt{t_3}|\alpha|)\sqrt{s_3} \\ & \leq \sqrt{(s_1 + t_4|\alpha|^2)(s_4 + t_1|\alpha|^2)} + \sqrt{(s_2 + t_3|\alpha|^2)(s_3 + t_2|\alpha|^2)} \end{aligned}$$

by the Cauchy-Schwartz inequality. \square

The converse of the last statement does not hold, as the examples in [19] show. Now, we have bunch of examples which distinguish various notions of positivity and their intersections. For example, if we take

$$\sqrt{s_1 t_1} = 0, \quad \sqrt{s_2 t_2} = 1, \quad \sqrt{s_3 t_3} = 1, \quad \sqrt{s_4 t_4} = 2, \quad |u_i| = 1, \quad i = 1, 2, 3, 4,$$

then the resulting map is $(1, 2, 2)$ -positive but neither $(2, 1, 2)$ nor $(2, 2, 1)$ -positive. If we take

$$\sqrt{s_1 t_1} = 0, \quad \sqrt{s_2 t_2} = 0, \quad \sqrt{s_3 t_3} = 2, \quad \sqrt{s_4 t_4} = 2, \quad |u_i| = 1, \quad i = 1, 2, 3, 4,$$

then the map is both $(1, 2, 2)$ and $(2, 1, 2)$ -positive but not $(2, 2, 1)$ -positive. If we put $\sqrt{s_i t_i} = 0$ for $i = 1, 2, 3$ and $\sqrt{s_4 t_4} = 4$, then we may find an example of $(1, 1, 1)$ -positive map which is not (p, q, r) -positive for other (p, q, r) . Finally, we may also get an example of (p, q, r) -positive map for each (p, q, r) in $\Sigma_{2,2,2}$ except for $(2, 2, 2)$ by putting $\sqrt{s_1 t_1} = 0$ and $\sqrt{s_i t_i} = 2$ for $i = 2, 3, 4$.

By Theorem 6.2 and Corollary 5.3, we have the following.

Corollary 6.4. *Suppose that $\phi : M_2 \times M_2 \rightarrow M_2$ is a bilinear map given by its Choi matrix as (12). Then the following are equivalent:*

- (i) $\langle \varrho, \phi \rangle \geq 0$ for each bi-separable three qubit state ϱ ;
- (ii) inequality (13) holds for each possible choice of i, j with $i \neq j$ from $\{1, 2, 3, 4\}$.

Therefore, if W is the Choi matrix of a bi-linear map ϕ satisfying the conditions in Corollary 6.4 and $\langle \varrho, \phi \rangle < 0$ then ϱ is a genuinely entangled state. In this sense, those W are genuine entanglement witnesses. We consider the following matrix

$$W = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 \\ \cdot & s & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & s & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & s & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & t & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & t & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & t & \cdot \\ -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix},$$

with $st = 1$. We note that this W is a genuine entanglement witness, since it satisfies the condition (ii) of Corollary 6.4. Consider the GHZ type pure state [1] given by the vector

$$|\psi_{\text{GHZ}}\rangle = \lambda_0|000\rangle + \lambda_1 e^{i\theta}|100\rangle + \lambda_2|101\rangle + \lambda_3|110\rangle + \lambda_4|111\rangle,$$

with $\lambda_i \geq 0$. Then we see that

$$\langle \psi_{\text{GHZ}} | \langle \psi_{\text{GHZ}} |, W \rangle = t(\lambda_1^2 + \lambda_2^2 + \lambda_3^2) - 2\lambda_0\lambda_4.$$

Taking arbitrary small $t > 0$, we see that W detects every $|\psi_{\text{GHZ}}\rangle \langle \psi_{\text{GHZ}}|$ with $\lambda_0\lambda_4 > 0$. These include GHZ type entangled pure states in the classification [1] of three qubit states.

7. DISCUSSION

We have defined in Section 4 the notion of Schmidt rank for the tensor product of arbitrary number of vector spaces. We also note that Theorem 5.1 is easily extended to $(n - 1)$ -linear maps and n -partite states in an obvious way, with this definition. This was also useful to clarify how the usual Schmidt rank for tensor of two spaces can be explained in our definition, and explain bi-separability in 3-partite cases. In the 4-partite case, it is possible with our definition to explain bi-separability according the bi-partition $4 = 1 + 3$ like A - BCD bi-separability. But we cannot explain bi-separability according the bi-partition $4 = 2 + 2$. It would be interesting to refine the notions of positivity and Schmidt numbers with which we may explain all kinds of bi-separability for the cases of $n \geq 4$.

As for (p, q, r) -positive bi-linear maps, we have shown that different triplets (p, q, r) in $\Sigma_{a,b,c}$ give rise to different convex cones $\mathbb{P}_{p,q,r}$, and we give concrete examples in 2×2 matrices. It would be interesting to give concrete examples in higher dimensions. Recall that the first examples which distinguish 2-positivity and 3-positivity in the 3×3 matrices was given by Choi [3]. See also [2].

We would like to remind the readers that we did not define complete positivity for bi-linear maps. It seems to be reasonable to say that a bi-linear map is completely positive when it is (∞, ∞, ∞) -positive, that is, (p, q, r) -positive for every triplet (p, q, r) , or equivalently, satisfies the condition (1) for each $p, q = 1, 2, \dots$. We note that the term ‘complete positivity’ for multi-linear maps already used in [7, 13] in totally different contexts from ours. Furthermore, the authors of [17] call those bi-linear maps with the condition (1) for each $p, q = 1, 2, \dots$ ‘jointly completely positive’ maps. As for similar problems in terminologies in the notions of complete boundedness for bi-linear maps, we refer to comments in [21].

Anyway, we call temporarily (∞, ∞, ∞) -positive bi-linear maps completely positive maps. Then we may define various kinds of complete copositivity and decomposability for bi-linear maps $\phi : M_A \times M_B \rightarrow M_C$ between matrix algebras. Recall that a linear map $\phi : M_A \rightarrow M_C$ is completely copositive if $\phi \circ t_A$ is completely positive. This is the case if and only if $t_C \circ \phi$ is completely positive, where t_A and t_C denote the transpose maps in M_A and M_C , respectively. There are three kinds of complete copositivity according to the complete positivity of the maps

$$\phi \circ (\text{id}_A \times t_B), \quad \phi \circ (t_A \times \text{id}_B), \quad t_C \circ \phi$$

for a bi-linear map $\phi : M_A \times M_B \rightarrow M_C$.

We say that a bi-linear map is decomposable if it is the sum of a completely positive map and three kinds of completely copositive maps. We would like to ask what kinds of (p, q, r) -positivity imply decomposability. As for bi-linear maps in Theorem 6.2, it is easy to see that they are decomposable whenever they are $(1, 2, 2)$, $(2, 1, 2)$ or $(2, 2, 1)$ -positive. For examples of indecomposable $(1, 1, 1)$ -positive bi-linear maps in M_2 , we refer to [19]. There is a long standing question which asks if every 2-positive linear map between M_3 is decomposable. See Corollary 4.3 in [2]. The dual question asks if every $3 \otimes 3$ PPT state has Schmidt number less than or equal to 2. This was conjectured in [23]. Finally, it would be interesting to define decomposability for general situations beyond matrix algebras, as in the case of linear maps. See [25].

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